Intelligens Számítási Módszerek
Fuzzy relációk,
szabály alapú Fuzzy rendszerek

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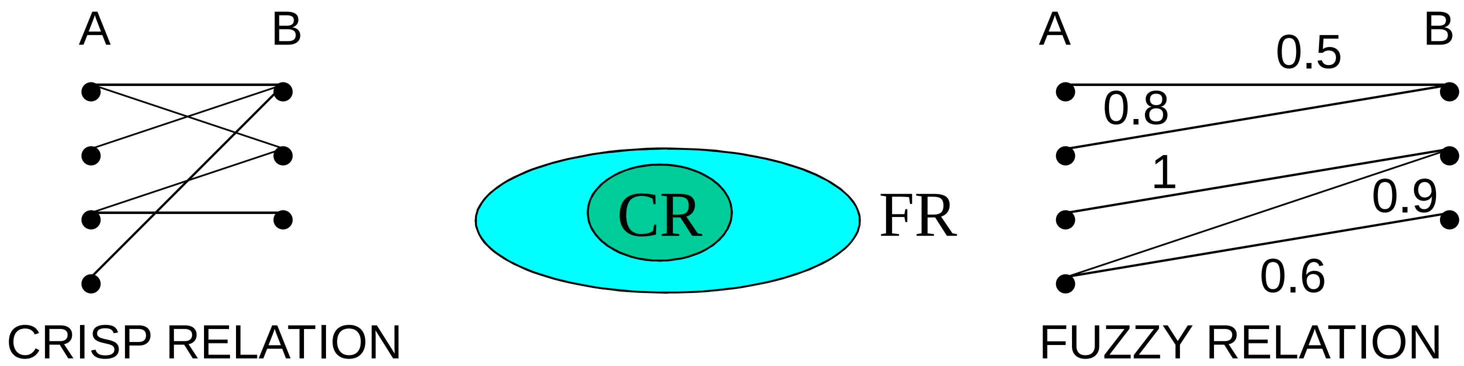
Általános INFORMATIKAI Tanszék

Crisp relation – Fuzzy relation

- **Crisp relation: Some interaction or association between elements of two or more sets.**
- **Fuzzy relation: Various degrees of association can be represented.**



- **Cartesian (direct) product of two (or more) sets**
  X, Y $\quad X \times Y = \{ (x,y) \mid x \in X, y \in Y \}$
  $X \times Y \neq Y \times X$ **IF** $X \neq Y$ **!**
- **More generally:** $$\mathop{\times}_{i=1}^{n} x_i = \left\{ \left(x_1, x_2, \ldots, x_n \right) \middle| x_i \in X_i, i \in N_n \right\}$$

Crisp relation

IF $X_i = x \quad \forall\, i \in N_n$ $\qquad X \times X \times \ldots \times X = X^n$

Crisp relation (mathematically) $\quad R(X_1, X_2, \ldots, X_n) \subset \underset{i \in N_n}{\times} X_i$

Characteristic function

$$\mu_R(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{IFF } (x_i) \in R \\ 0 & \text{ELSE} \end{cases}$$

LANGUAGE: L = { CHINESE, KOREAN, JAPANESE, ENGLISH }
COUNTRY: C = { KOREA, CHINA, TAIWAN, JAPAN, HONGKONG }
GEOGRAPHY: G = { MAINLAND, ISLAND }

R(L,C,G):

MAINLAND

$$\begin{array}{l} \\ \text{CHINESE} \\ \text{KOREAN} \\ \text{JAPANESE} \\ \text{ENGLISH} \end{array} \begin{array}{c} \begin{matrix} K & C & T & J & H \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

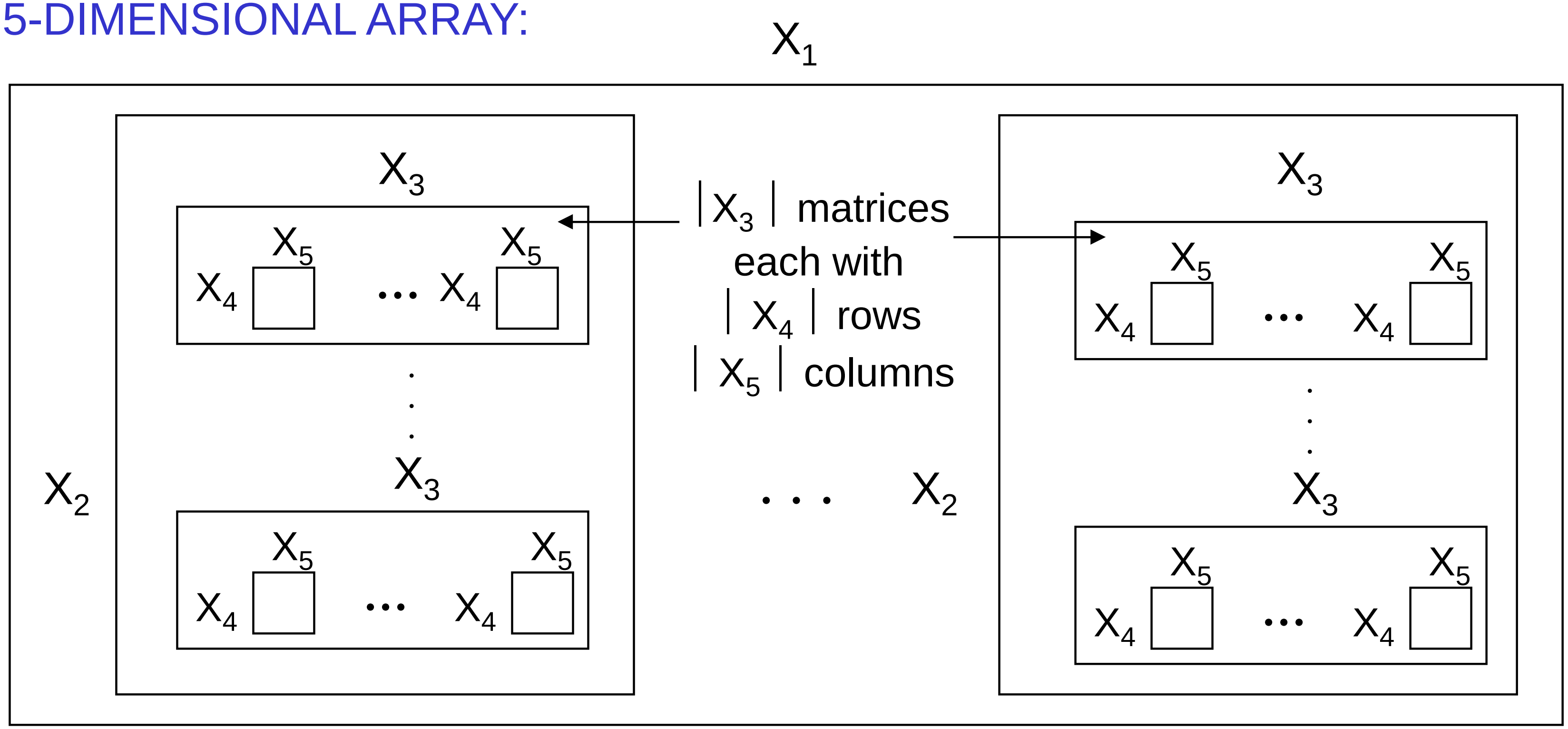
ISLAND

$$\begin{array}{l} \\ \text{CHINESE} \\ \text{KOREAN} \\ \text{JAPANESE} \\ \text{ENGLISH} \end{array} \begin{array}{c} \begin{matrix} K & C & T & J & H \end{matrix} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Crisp relation

MATRIX REPRESENTATION OF n-ARY RELATIONS:

A POSSIBLE REPRESENTATION OF QUINARY RELATION BY 5-DIMENSIONAL ARRAY:



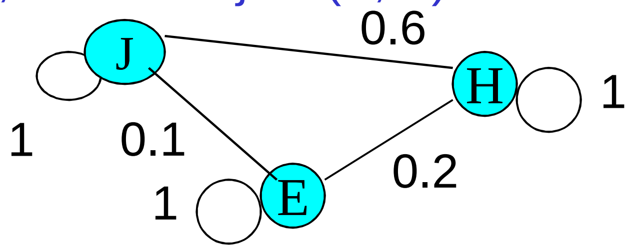
Fuzzy relation

Fuzzy relation: { 0, 1 } **is extended to** [0, 1]

SIMILARITY OF LANGUAGES

X= { JAPANESE, HUNGARIAN, ENGLISH} : R(X, X)

$$\begin{array}{c} \\ J \\ H \\ E \end{array} \begin{array}{c} \begin{array}{ccc} J & H & E \end{array} \\ \begin{pmatrix} 1 & 0.6 & 0.1 \\ 0.6 & 1 & 0.2 \\ 0.1 & 0.2 & 1 \end{pmatrix} \end{array}$$



Fuzzy relation - definitions

Subsequence:

$$\underline{x} = (x_i \mid i \in N_n) \in \mathop{\times}_{i \in N_n} X_i \quad , \quad \underline{y} = (y_j \mid j \in J) \in \mathop{\times}_{j \in J} X_j \quad , \quad J \subset N_n$$

$\underline{y}$ is a subsequence of $\underline{x}$ IFF $\forall\, j \in J : y_j = x_j$

$\underline{y} \prec \underline{x}$ **(notation)**

Fuzzy relation - definitions

Projection of a relation:

$R \downarrow Y$ **projection to** Y

$$\mu_{R \downarrow Y}(\underline{y}) = \max_{y \prec x} \left(\mu_R(\underline{x}) \right)$$

$X = \{ x, y \}$ $A = \{ +, * \}$ $Q = \{ \$, £ \}$

$R(x, a, q) = 0.1 / (x, +, \$) + 0.3 / (x, +, £) + 0.4 / (x, *, \$) + 0.8 / (y, +, £) + 1 / (y, *, \$)$

$R_{XA} = R \downarrow (X \times A)$

$R_{XA}(x, a) = 0.3 / (x, +) + 0.4 / (x, *) + 0.8 / (y, +) + 1 / (y, *)$

$R_{XQ} = R \downarrow (X \times Q)$

$R_{XQ}(x, q) = 0.4 / (x, \$) + 0.3 / (x, £) + 1 / (y, \$) + 0.8 / (y, £)$

$R_X = R \downarrow X$

$R_X(x) = 0.4 / x + 1 / y$

Projection of a relation

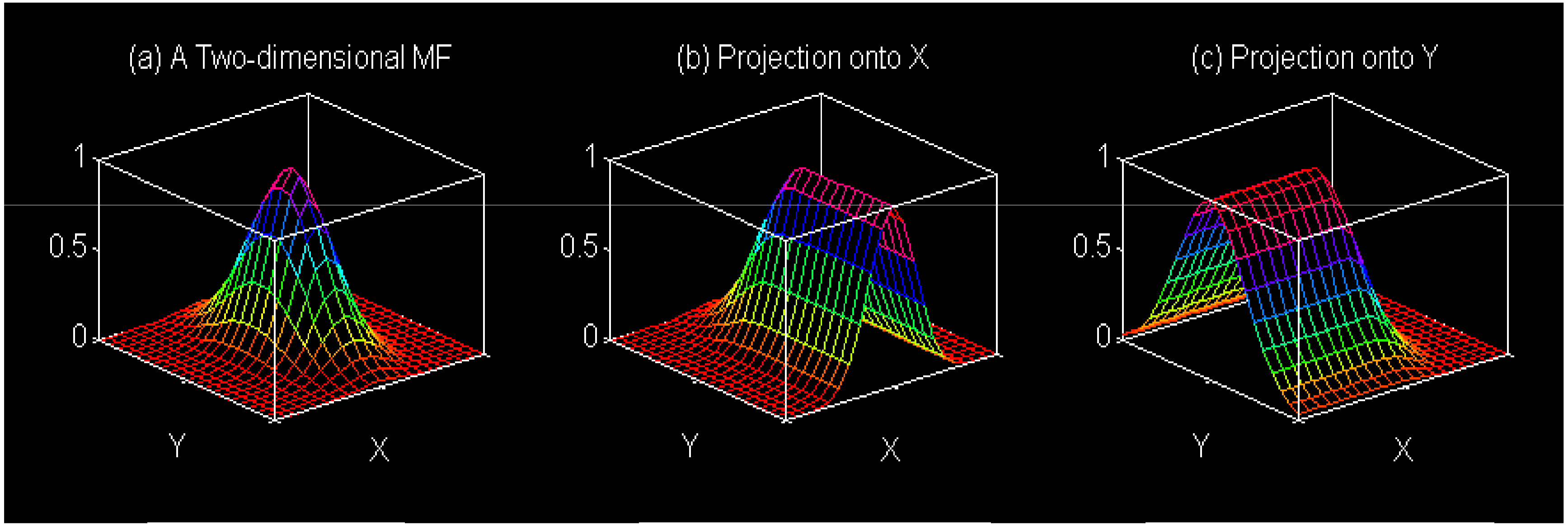
Two-dimensional MF

Projection onto X

$R_X = R \downarrow X$

Projection onto Y

$R_Y = R \downarrow Y$



$$\mu_R(x, y)$$

$$\mu_A(x) = \max_y \mu_R(x, y)$$

$$\mu_B(y) = \max_x \mu_R(x, y)$$

Fuzzy relation - definitions

Cylindric extension

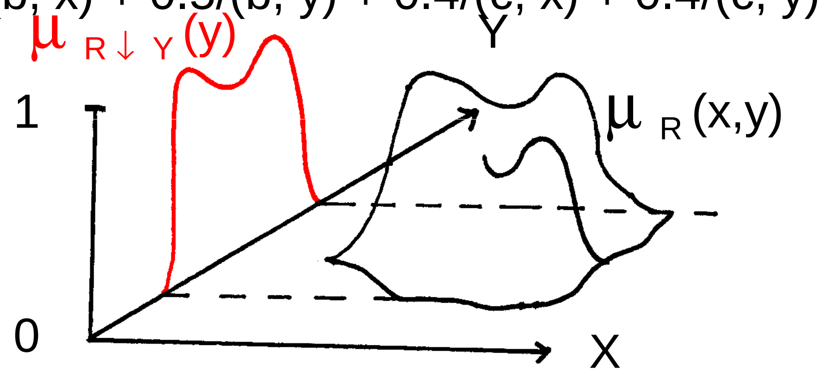
R $\uparrow$ (X-Y) **cylindric extension to** X

Y $\subset$ X **where** R(Y) **was defined**

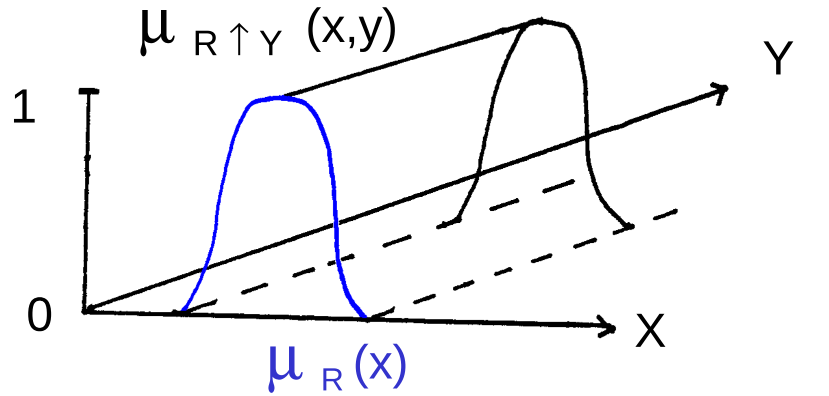
$$\mu_{R\uparrow(X-Y)}(\underline{x}) = \mu_R(\underline{y})$$
$$x \succ y$$

Y = { a, b, c } , $\mu_R(y) = 0.3 / a + 0.5 / b + 0.4 / c$, X = { a, b, c } $\times$ { x, y }

$$\mu_{R\uparrow(X-Y)}(\underline{x}) = 0.3/(a, x) + 0.3/(a, y) + 0.5/(b, x) + 0.5/(b, y) + 0.4/(c, x) + 0.4/(c, y)$$



$$\text{supp}(R \downarrow Y) = \max_{x} (\text{supp}(R))$$

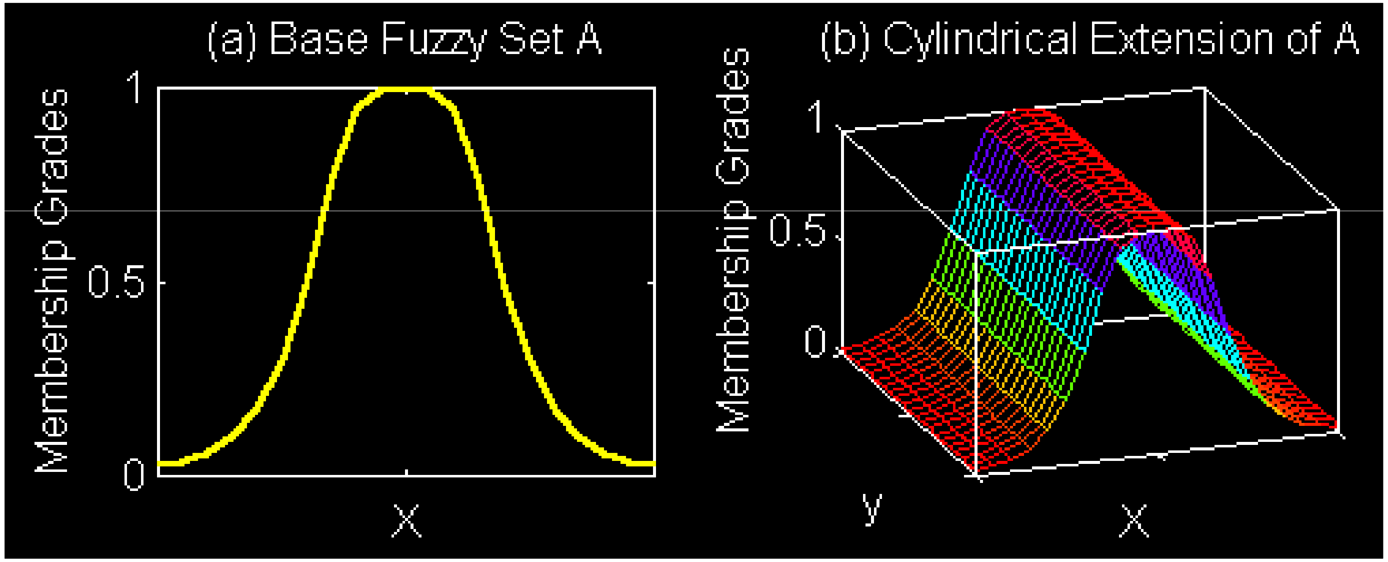


$$\text{supp}(R \uparrow Y) = \text{supp}(R) \times Y$$

Cylindrical extension

Base set A **Cylindrical Extension of A**

$R_{X,Y} = R \uparrow Y$



Fuzzy relation - definitions

Cylindric closure:

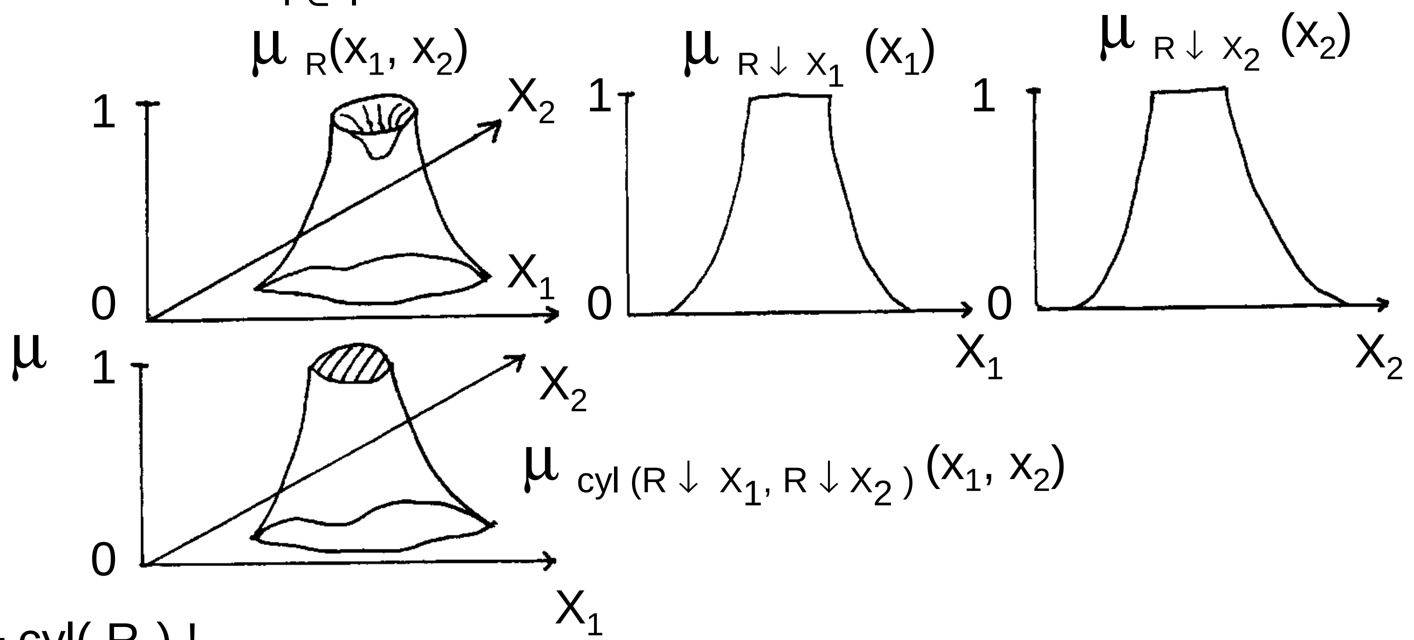
$R(x_1, x_2, \dots, x_n)$ **is not known**

Known are $\mu_{R\downarrow Y_1}, \mu_{R\downarrow Y_2}, \dots \quad i \in I$

$$Y_i = \bigtimes_{j \in J_i} X_j \qquad i \in I \qquad\qquad \bigtimes_{i \in I} Y_i = \bigtimes_{i=1}^{n} X_j$$

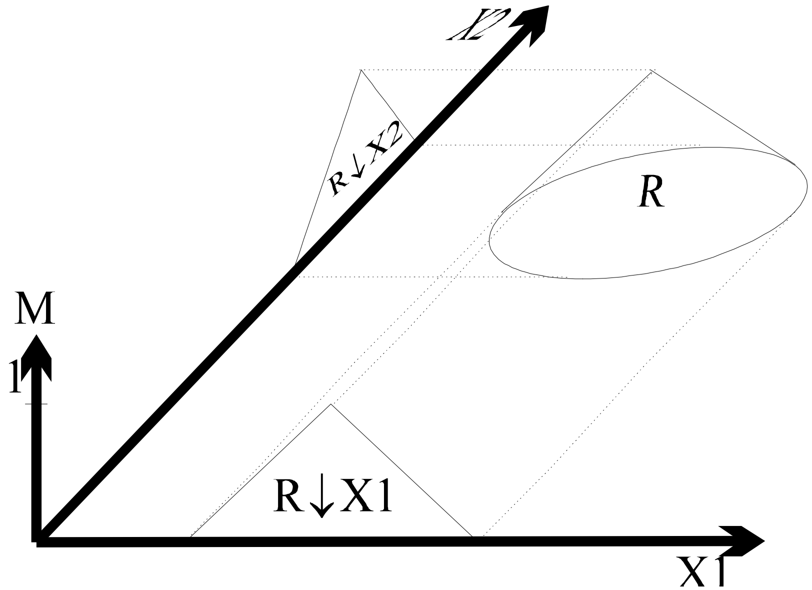
Then $cyl(R\downarrow Y_i)$ **approximates** R

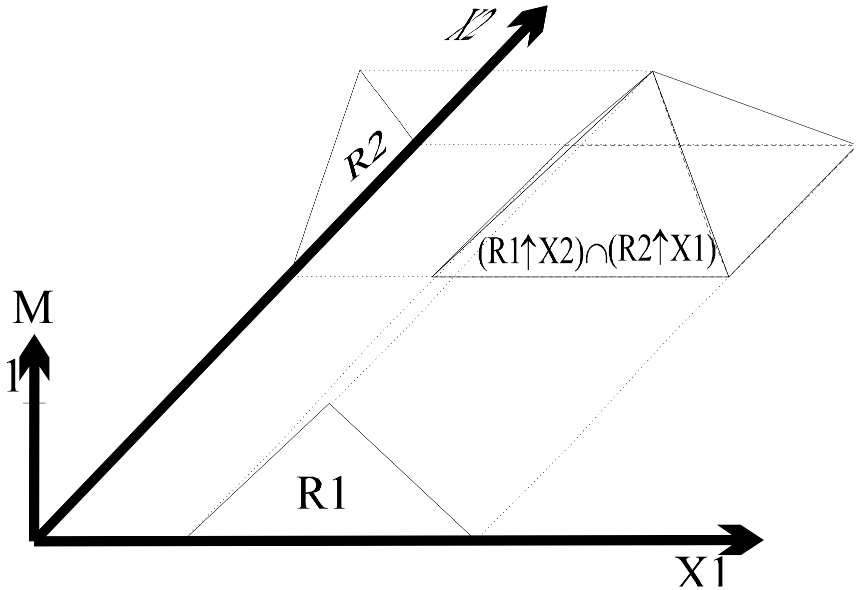
$$\mu_{cyl(R\downarrow Y_i)}(\underline{x}) = \min_{i \in I} \left(\mu_{(R\downarrow Y_i)\uparrow X - Y_i}(\underline{x}) \right)$$



$R \neq cyl(R_i)$!

Cylindric closure



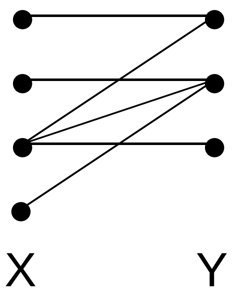


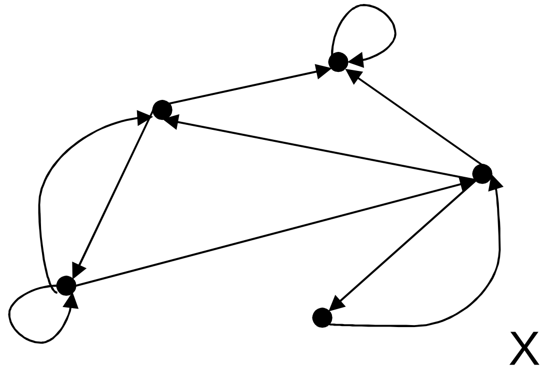
Binary relation

Binary relation: Relation between two sets (X, Y)
R (X,Y)

If X ≠ Y **Binary relation** = **Bipartite graph**

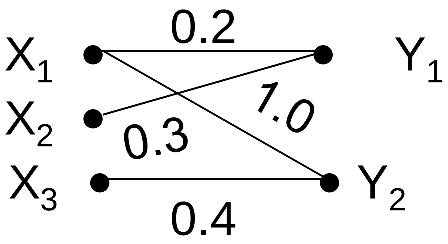
If X = Y **Directed graph (Digraph)**

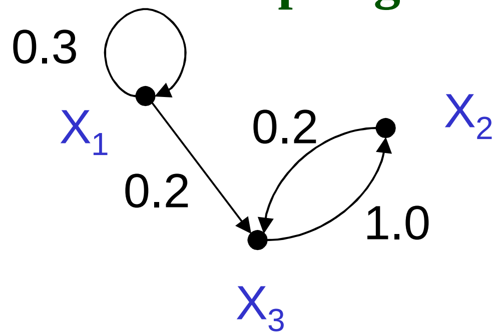




Fuzzy binary relation

Fuzzy binary relation: **Every edge bears a membership degree**





(**Sagittal diagram**)

$$\begin{array}{c|cc} & Y_1 & Y_2 \\ \hline X_1 & 0.2 & 1.0 \\ X_2 & 0.3 & 0.0 \\ X_3 & 0.0 & 0.4 \end{array}$$

$$\begin{array}{c|ccc} & X_1 & X_2 & X_3 \\ \hline X_1 & 0.3 & 0.0 & 0.2 \\ X_2 & 0.0 & 0.0 & 0.2 \\ X_3 & 0.0 & 1.0 & 0.0 \end{array}$$

Denotation:

$R(x,y)$ $\quad x R y \quad$ (CF. $x < y$)

Fuzzy case:

$\mu_R(x,y) \quad \alpha / xRy \;\hat{=}\; \mu_R(x,y) = \alpha$

Fuzzy binary relation

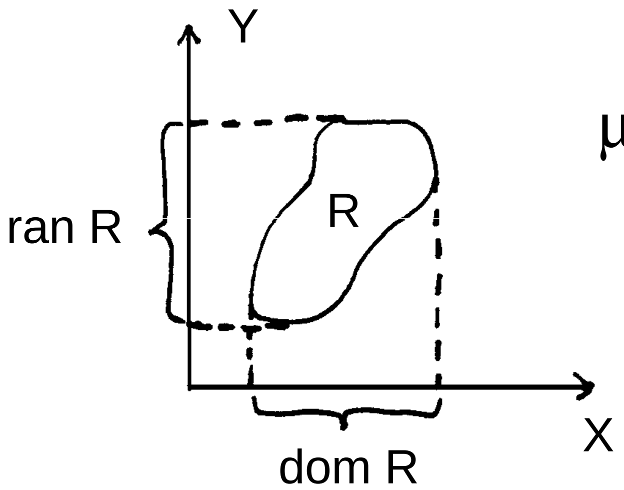
Domain and Range

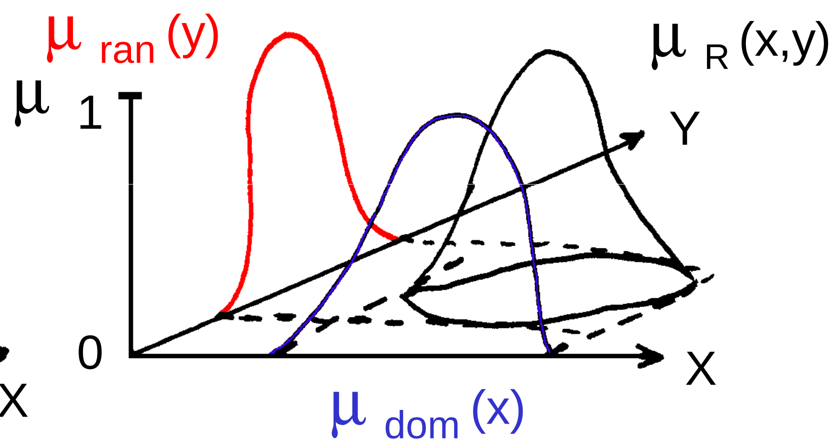
dom R (X,Y) = { x | x ∈ X, (x, y) ∈ R   ∃ y ∈ Y }

ran R (X,Y) = { y | y ∈ Y, (x, y) ∈ R   ∃ x ∈ X }

Fuzzy Domain and Range

$$\mu_{\text{dom } R}(x) = \max_{y \in Y} \mu_R(x,y) \qquad \mu_{\text{ran } R}(y) = \max_{x \in X} \mu_R(x,y)$$





Domain and Range are projections of R

Height of R:

$$h(R) = \max_{x \in X} \max_{y \in Y} \mu_R(x,y) = h(\text{dom } R) = h(\text{ran } R)$$

Classification of binary relations

1) COMPLETELY SPECIFIED: $\mathrm{dom}\, R = X$

2) INCOMPLETELY SPECIFIED: $\mathrm{dom}\, R \neq X$

3) ONTO RELATION: $\mathrm{ran}\, R = Y$

4) INTO RELATION: $\mathrm{ran}\, R \neq Y$

5) MAPPING (FUNCTION): $R\,(X \rightarrow Y)$

$(\forall\, x \mid x \in X)\,(\nexists\; y_1, y_2 \mid y_1 \neq y_2 ;\; y_1, y_2 \in Y)\;(xRy_1 \text{ AND } xRy_2)$

ONLY ONE ‘IMAGE’

6) ONE TO MANY:

$(\exists\, x \mid x \in X)\,(\exists\; y_1, y_2 \mid y_1 \neq y_2 ;\; y_1, y_2 \in Y)\;(xRy_1 \text{ AND } xRy_2)$

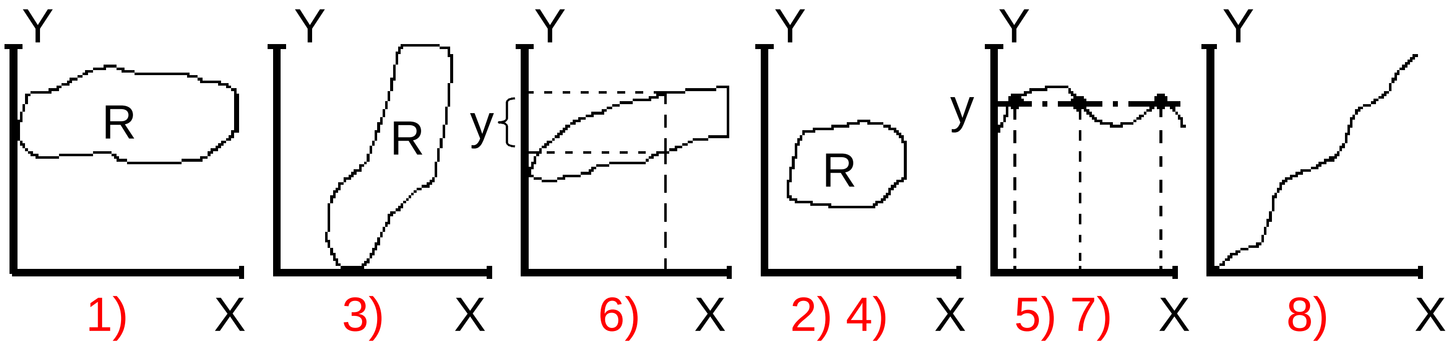
7) MANY TO ONE: R IS MAPPING AND

$(\exists\, y \mid y \in Y)\,(\exists\; x_1, x_2 \mid x_1 \neq x_2 ;\; x_1, x_2 \in X)\;(x_1Ry \text{ AND } x_2Ry)$

8) ONE TO ONE:

$(\forall\, x \mid x \in X)\,(\forall\, y \mid y \in Y)$

$(xRy) \rightarrow [(\nexists\, x' \mid x' \neq x, x' \in X)\,(x'Ry) \text{ AND } (\nexists\, y' \mid y' \neq y, y' \in Y)\,(xRy')\,]$



Resolution form of a fuzzy relation

Resolution form (based on α-cuts)

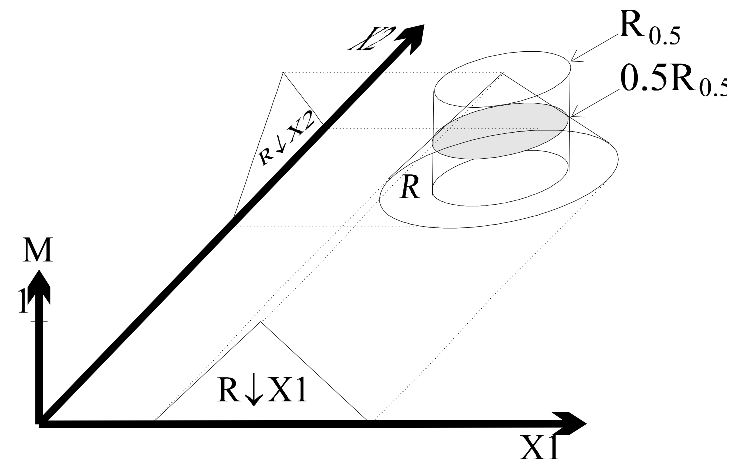
$$R = \bigcup_{\alpha} \alpha R_{\alpha} \quad \alpha \in \Lambda_R$$

max

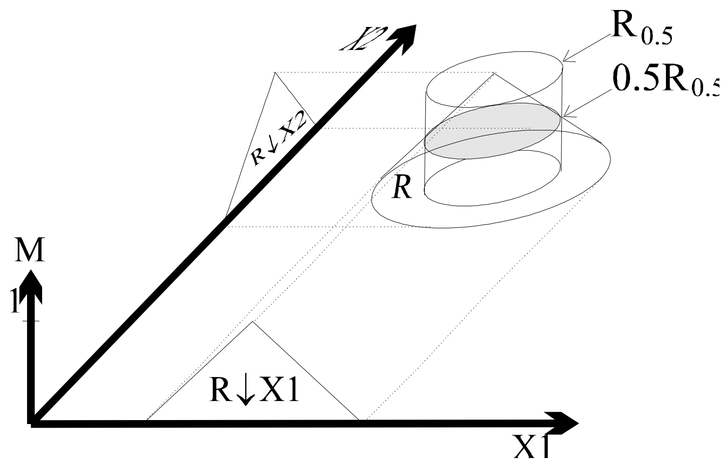
(Level set)

Characteristic function

$$\mu_{\alpha R_{\alpha}}(x,y) = \alpha \, \mu_{R_{\alpha}}(x,y)$$



Resolution form of a fuzzy relation



$X = \{ x_1, x_2, x_3 \} \qquad R\,(X, X)$

$$\underline{\underline{M}}_R = \begin{pmatrix} 0.6 & 0.0 & 1.0 \\ 0.4 & 0.3 & 0.0 \\ 0.5 & 0.2 & 0.3 \end{pmatrix}$$

$\Lambda_R = \{ 0.0, 0.2, 0.3, 0.4, 0.5, 0.6, 1.0 \}$

$\alpha = 0$ **can be ignored** $0\mu_X = \mu_0$

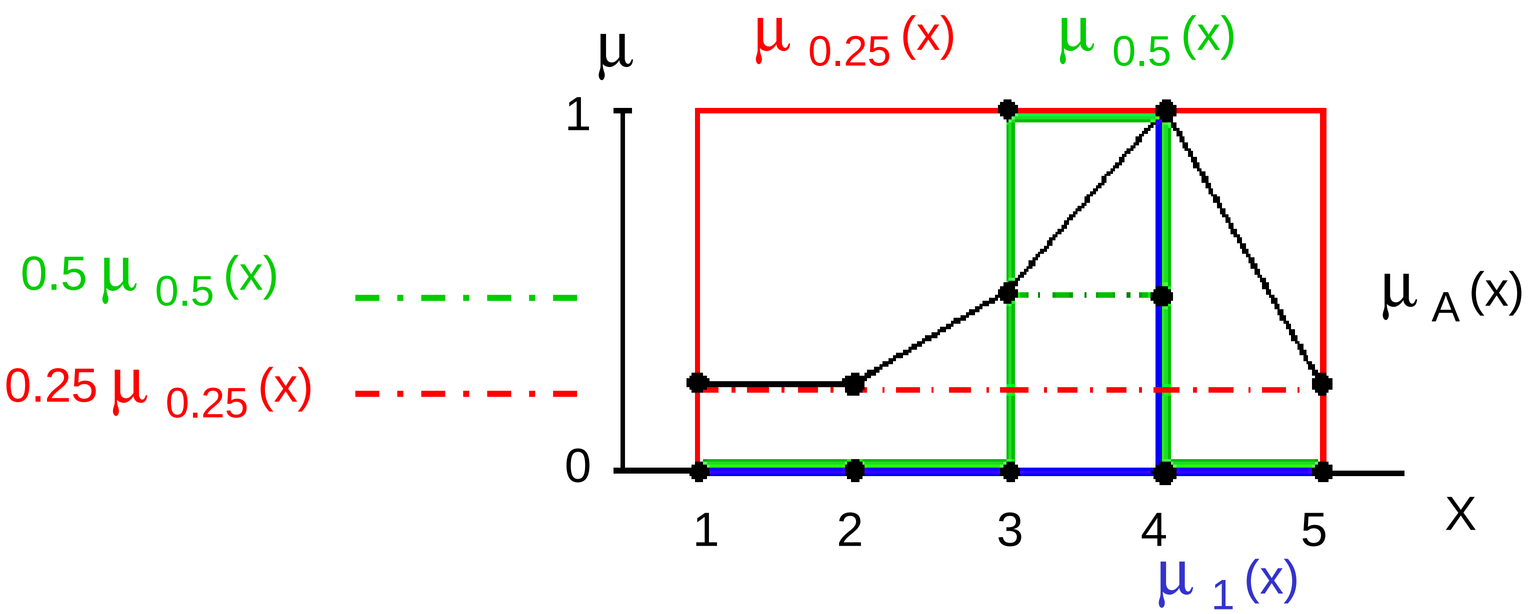
$$\underline{\underline{M}}_R = 0.2 \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} + 0.3 \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + 0.4 \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} +$$

$$+\, 0.5 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + 0.6 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 1.0 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Resolution form of a fuzzy relation

This technique is more generally used: **Resolution principle**

$X = \{ 1, 2, 3, 4, 5 \}$ $\Lambda_A = \{ 0.25, 0.5, 1 \}$



Fuzzy binary relation - inverse

Inverse of a binary relation:

$R^{-1}(\mathbf{Y}, \mathbf{X}) = \{ (y, x) \mid (x, y) \in R \}$

$\mathrm{dom}R^{-1} = \mathrm{ran}R, \quad \mathrm{ran}R^{-1} = \mathrm{dom}R$

In the fuzzy case:

$\mu_{R^{-1}}(y, x) = \mu_R(x, y) \qquad (x, y) \in X \times Y$

$\underline{\underline{M}}_{R^{-1}} = \underline{\underline{M}}_R^T$

$(R^{-1})^{-1} = R$

$X = \{ x_1, x_2, x_3 \}$

$Y = \{ y_1, y_2 \}$

$$\underline{\underline{M}}_R = \begin{bmatrix} 0.3 & 0.0 \\ 0.5 & 0.6 \\ 0.8 & 1.0 \end{bmatrix} \Rightarrow \underline{\underline{M}}_{R^{-1}} = \begin{bmatrix} 0.3 & 0.5 & 0.8 \\ 0.0 & 0.6 & 1.0 \end{bmatrix}$$

Crisp relation - composition

Composition of relations (crisp)

$P(X, Y)$, $Q(Y, Z)$ **two crisp relations**

$R(X, Z) = P(X, Y) \circ Q(Y, Z)$

$R(X, Z) \subset X \times Z$

$(x, z) \in R$ **IFF** $\left(\exists\, y \mid y \in Y \right) \left((x, y) \in P \right.$ **AND** $\left. (y, z) \in Q \right)$

$P \circ Q \neq Q \circ P$

$(P \circ Q)^{-1} = Q^{-1} \circ P^{-1}$

$(P \circ Q) \circ R = P \circ (Q \circ R) = P \circ Q \circ R$

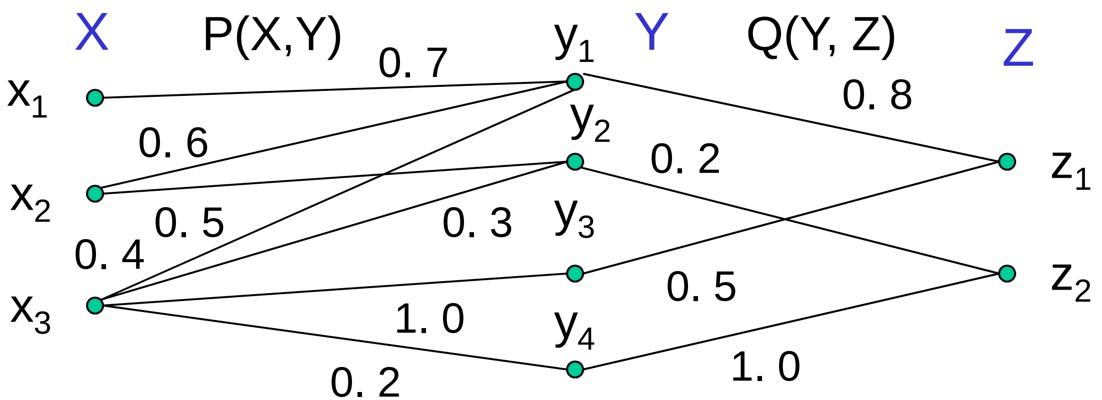
Composition of binary Fuzzy relations

Max-min composition of Fuzzy relations

$P(X,Y) : \mu_P \qquad Q(Y, Z) : \mu_Q$

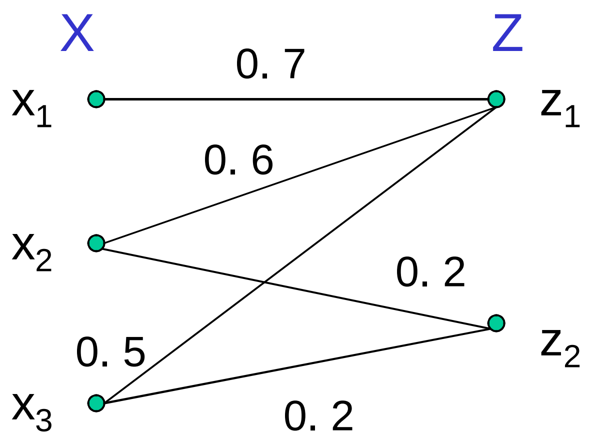
$$\mu_{P \circ Q}(x, z) = \max_{y \in Y} \min \left(\mu_P(x, y), \mu_Q(y, z) \right)$$

All properties for crisp relations are satisfied



$x_1 \ z_1$ min (0.7, 0.8) = 0.7
$x_1 \ z_2$ 0.0
$x_2 \ z_1$ min (0.6, 0.8) = 0.6
$x_2 \ z_2$ min (0.5, 0.2) = 0.2
$x_3 \ z_1$ min (0.4, 0.8) = 0.4
min (1.0, 0.5) = 0.5 max = 0.5
$x_3 \ z_2$ min (0.3, 0.2) = 0.2
min (0.2, 1.0) = 0.2 max = 0.2

$R = P \circ Q$



Other compositions of Fuzzy relations

Max-product composition:

$$\mu_{P \odot Q}(x,z) = \max_{y \in Y}\left(\mu_P(x,y) \cdot \mu_Q(y,z)\right)$$

General s-t composition: s = UNION, t = INTERSECTION

$$\mu_{P \circ_{s,t} Q}(x,z) = \underset{y \in Y}{S}\left(\mu_P(x,y) \, ⓣ \, \mu_Q(y,z)\right)$$

Composition of membership matrices

$\underline{\underline{M}}_P = [\, p_{ik} \,]$ $\underline{\underline{M}}_Q = [\, q_{kj} \,]$ $\underline{\underline{M}}_R = [\, r_{ij} \,]$

$$[\, r_{ij} \,] = [\, p_{ik} \,] \circ_{s,t} [\, q_{kj} \,]$$

$$r_{ij} = \underset{K}{S}\left(p_{ik} \, ⓣ \, q_{kj}\right)$$

Composition of membership matrices

$\underline{\underline{M}}_P = [\, p_{ik} \,]$ $\qquad$ $\underline{\underline{M}}_Q = [\, q_{kj} \,]$ $\qquad$ $\underline{\underline{M}}_R = [\, r_{ij} \,]$

$$[\, r_{ij} \,] = [\, p_{ik} \,] \circ_{s,t} [\, q_{kj} \,]$$

$$r_{ij} = \underset{K}{S} \left(p_{ik} \, ⓣ \, q_{kj} \right)$$

Example: **Algebric composition:** s = a + b - ab, $\quad$ t = ab

$X = \{ x_1, x_2 \}$ $\qquad$ $Y = \{ y_1, y_2 \}$ $\qquad$ $Z = \{ z_1, z_2 \}$

$$\underline{\underline{M}}_P = \begin{array}{cc} & Y \\ X & \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.4 \end{bmatrix} \end{array} \qquad \underline{\underline{M}}_Q = \begin{array}{cc} & Z \\ Y & \begin{bmatrix} 0.5 & 0.7 \\ 0.6 & 0.8 \end{bmatrix} \end{array} \qquad \underline{\underline{M}}_R = \underline{\underline{M}}_{P \circ_A Q} \cong \begin{array}{cc} & Z \\ X & \begin{bmatrix} 0.22 & 0.29 \\ 0.32 & 0.42 \end{bmatrix} \end{array}$$

$$(0.1 \cdot 0.5) + (0.3 \cdot 0.6) - (0.1 \cdot 0.5) \cdot (0.3 \cdot 0.6) = 0.221$$

Relational Join of binary Fuzzy relations

Relational join (of binary relations)

$P(X,Y) * Q(Y, Z) = \{ (x, y, z) \mid (x, y) \in P \text{ AND } (y, z) \in Q \}$ **(crisp case)**

$\mu_{P * Q}(x, y, z) = \min(\mu_P(x, y), \mu_Q(y, z))$ **(Fuzzy case)**

$\Rightarrow$ Trenary relation

Connection of $\circ$ and $*$:

$$\mu_{P \circ Q}(x, z) = \max_y \mu_{P * Q}(x, y, z)$$

(There is no possibility to determine $*$ from $\circ$)

Relational Join of binary Fuzzy relations

Example:

$X = \{ x_1, x_2 \} \qquad Y = \{ y_1, y_2 \} \qquad Z = \{ z_1, z_2 \}$

$P(X,Y) = 0.1 / (x_1, y_1) + 0.5 / (x_1, y_2) + 0.3 / (x_2, y_2)$

$Q(Y, Z) = 0.4 / (y_1, z_1) + 1.0 / (y_1, z_2) + 0.8 / (y_2, z_1)$

$$R(X,Y, Z) = P * Q = 0.1 / (x_1, y_1, z_1) + 0.1 / (x_1, y_1, z_2) + + 0.5 / (x_1, y_2, z_1) + 0.3 / (x_2, y_2, z_1)$$

$S(X, Z) = P \circ Q = 0.5 / (x_1, z_1) + 0.1 / (x_1, z_2) + 0.3 / (x_2, z_1)$

Other joins: min $\rightarrow$ t **e.g. product**

Max-min composition of Fuzzy relations

- The max-min composition of two fuzzy relations R_1 (defined on X and Y) and R_2 (defined on Y and Z) is

$$\mu_{R_1 \circ R_2}(x,z) = \bigvee_y [\mu_{R_1}(x,y) \wedge \mu_{R_2}(y,z)]$$

- **Properties**:
  - **Associativity:** $R \circ (S \circ T) = (R \circ S) \circ T$
  - **Distributivity over union:**

    $$R \circ (S \bigcup T) = (R \circ S) \bigcup (R \circ T)$$

  - **Week distributivity over intersection:**

    $$R \circ (S \bigcap T) \subseteq (R \circ S) \bigcap (R \circ T)$$

  - **Monotonicity:**

    $$S \subseteq T \Rightarrow (R \circ S) \subseteq (R \circ T)$$

Fuzzy inference systems

- **fuzzy rule based system**
- **fuzzy expert system**
- **fuzzy model**
- **fuzzy associative memory**
- **fuzzy logic controller**
- **fuzzy system**

Fuzzy inference

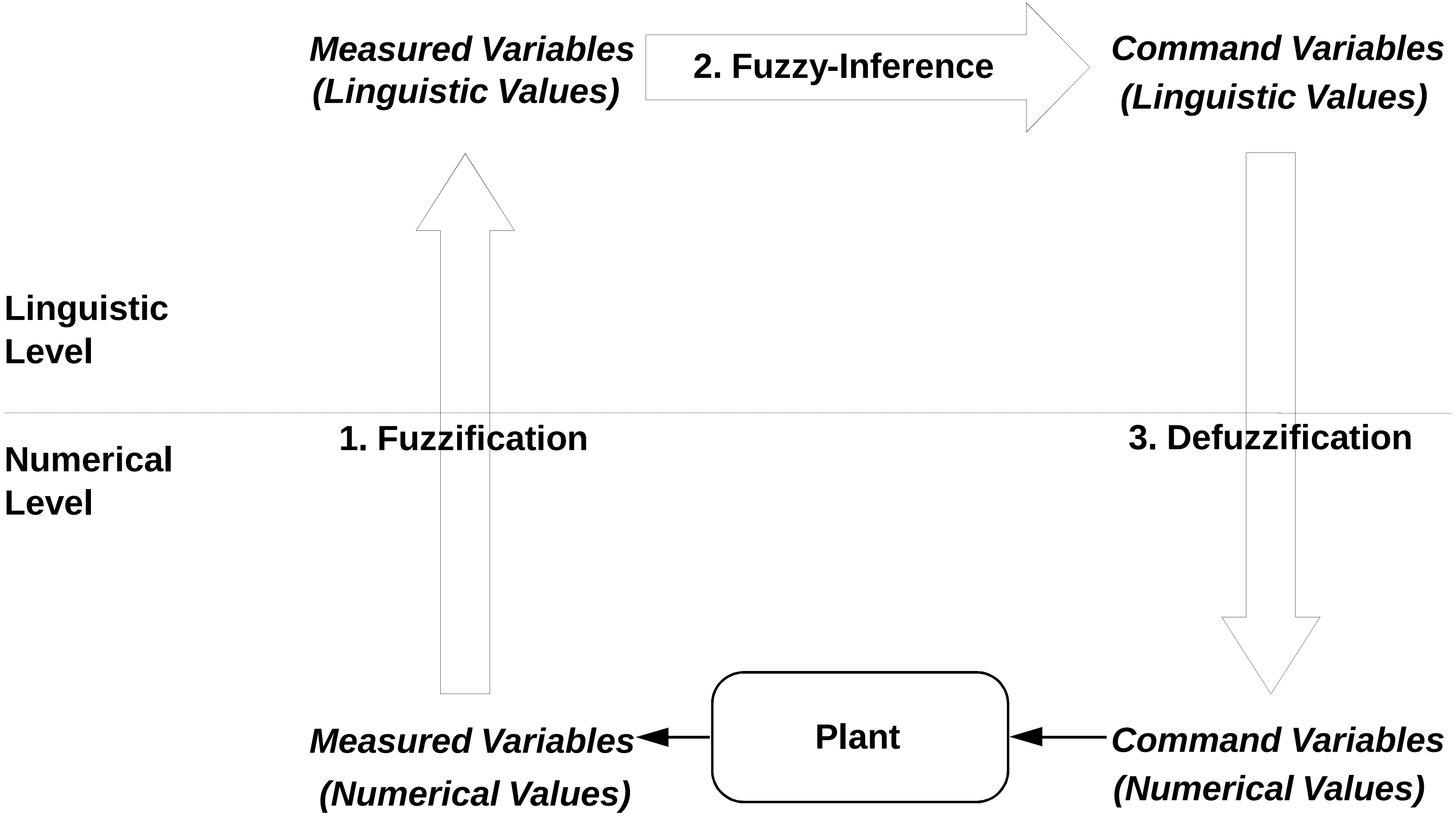
- **The most commonly used fuzzy inference technique is the so-called Mamdani method.**
- **In 1975, Professor Ebrahim Mamdani of London University built one of the first fuzzy systems to control a steam engine and boiler combination.**
- **He applied a set of fuzzy rules supplied by experienced human operators.**

Mamdani Fuzzy inference

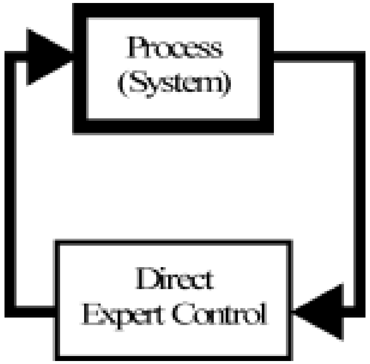
- **The Mamdani-style fuzzy inference process is performed in the following steps:**
  - **fuzzification of the input variables,**
  - **Fuzzy inference,**
  - **defuzzification.**
- **Inference mechanism applied:**

  Max-min compositional rule of inference (Zadeh)

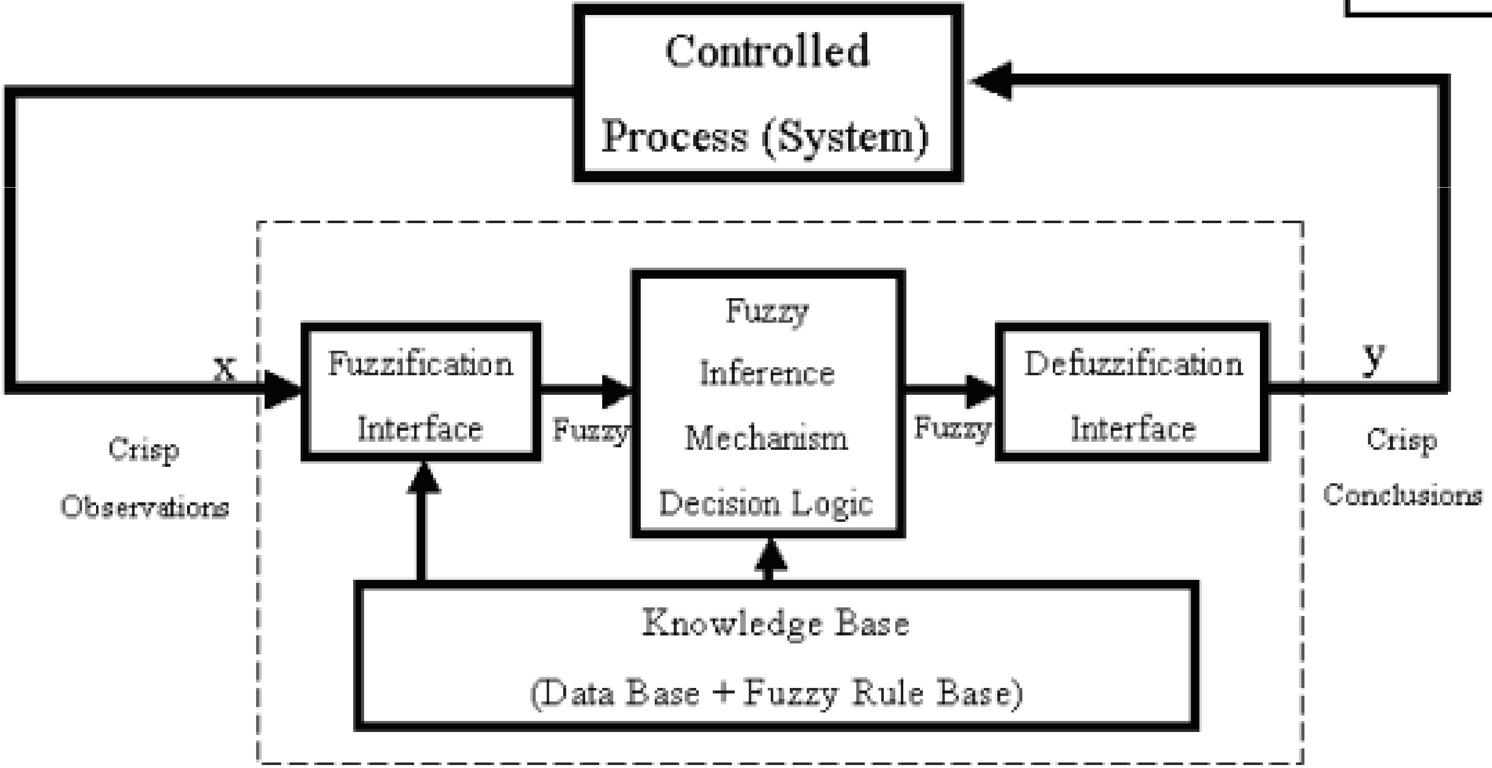
Fuzzification – Inference – Defuzzification



Direct Fuzzy Logic Control



Mamdani Model:



Fuzzy inference system

Three main components:

- **rule base**
- **data base (defines membership functions)**
- **reasoning mechanism (aggregation)**

Fuzzy If-Then Rules

- **General format:**

  If x is A then y is B

- **Examples:**
  - **If pressure is high, then volume is small.**
  - **If the road is slippery, then driving is dangerous.**
  - **If a tomato is red, then it is ripe.**
  - **If the speed is high, then apply the brake a little.**

Linguistic Variables

- A numerical variables takes numerical values:

  Age = 65

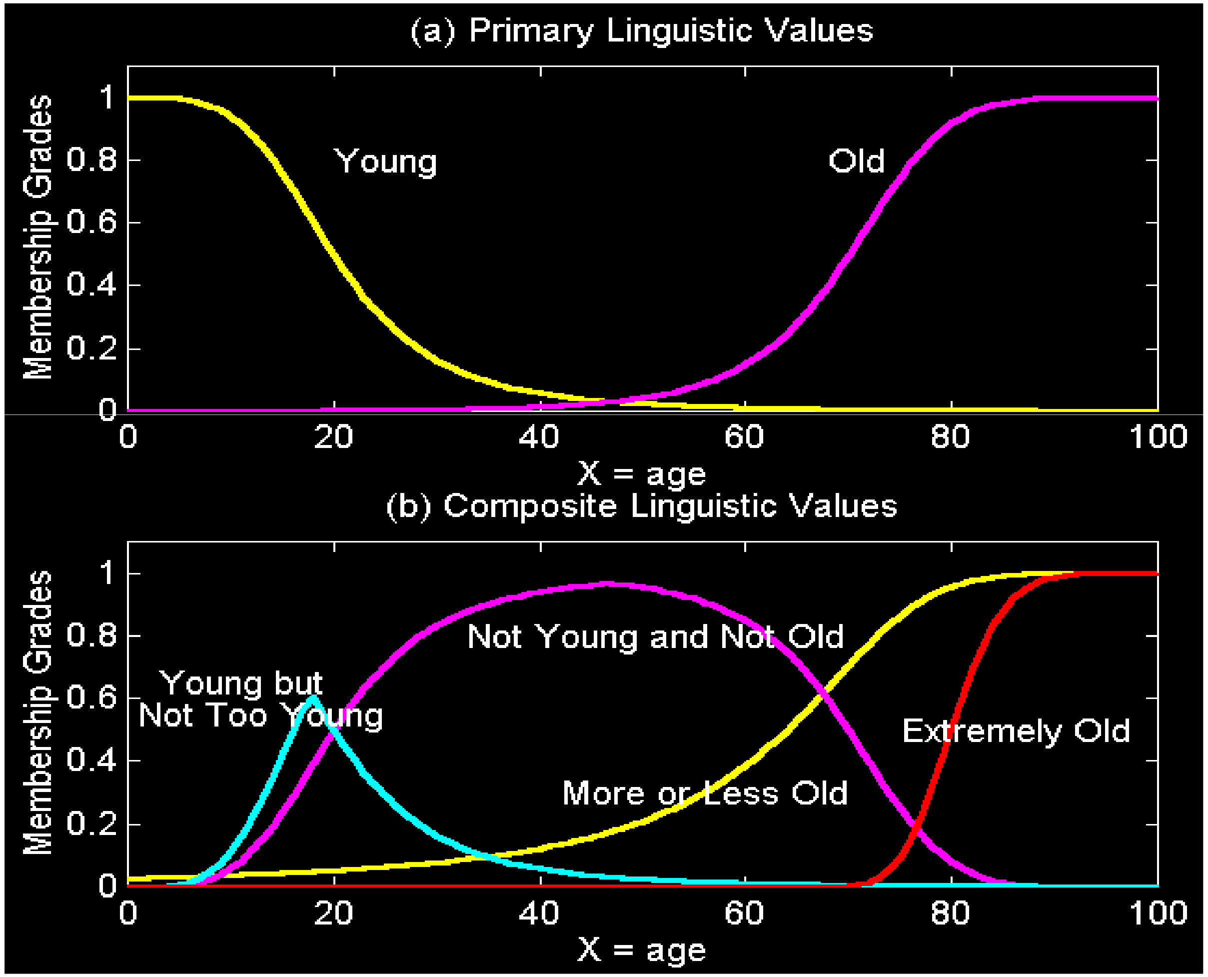
- A linguistic variables takes linguistic values:

  Age is old

- Linguistic values are fuzzy sets.
- All linguistic values form a term set:

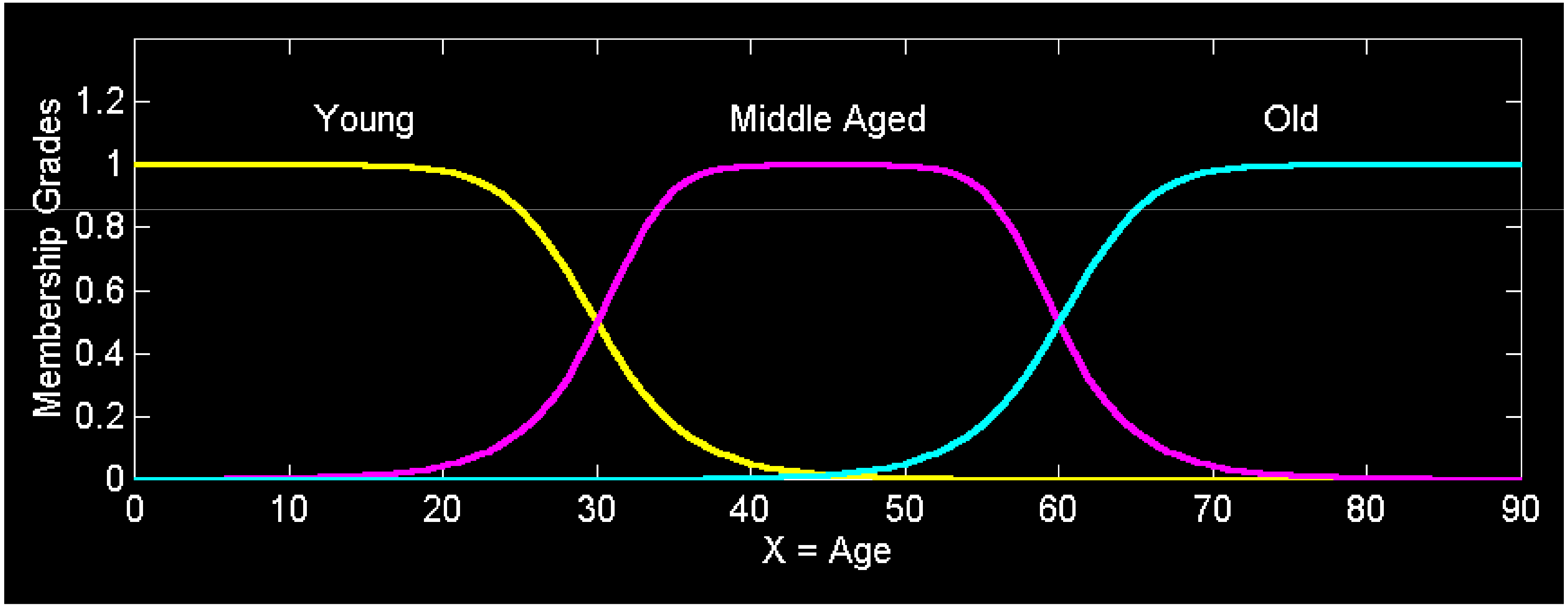
  T(age) = {young, not young, very young, ...
  middle aged, not middle aged, ...
  old, not old, very old, more or less old, ...
  not very yound and not very old, ...}

Linguistic Values (Terms)



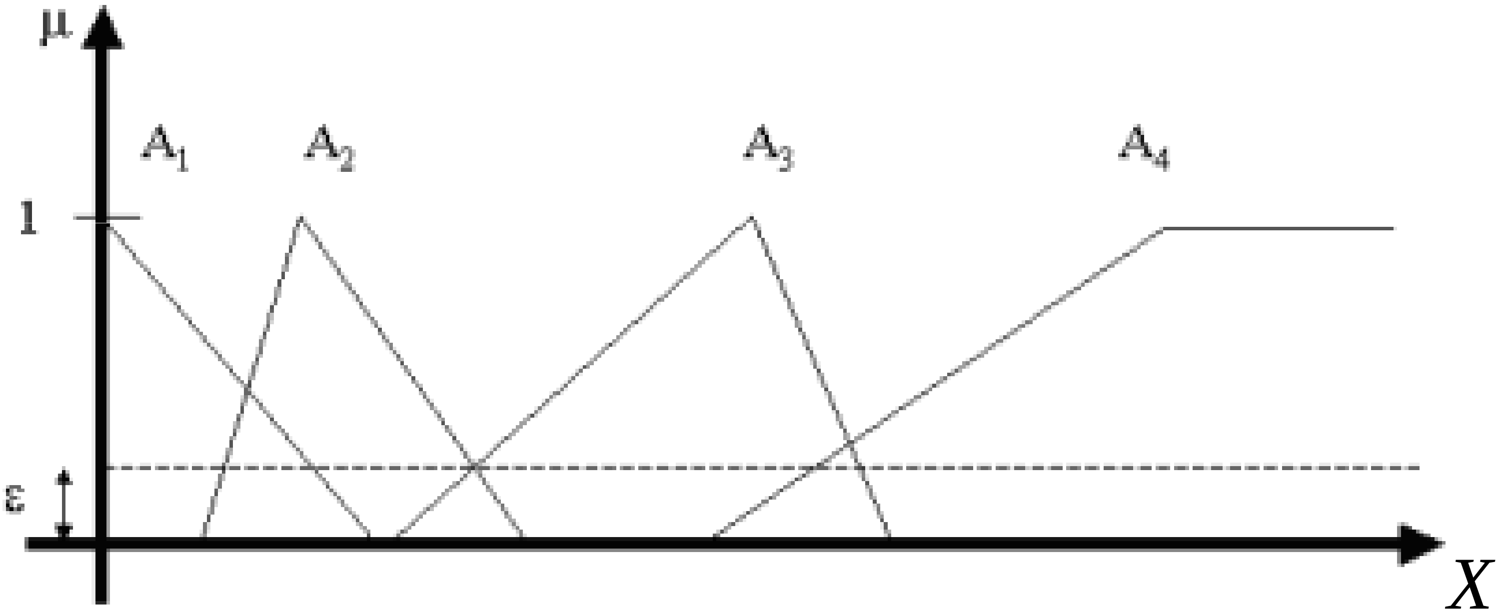
Fuzzy Partition

- **Fuzzy partitions formed by the linguistic values "young", "middle aged", and "old":**



ε-covering Fuzzy partition

- **The fuzzy partition (frame of cognition) ε-covers the universe of discourse X**

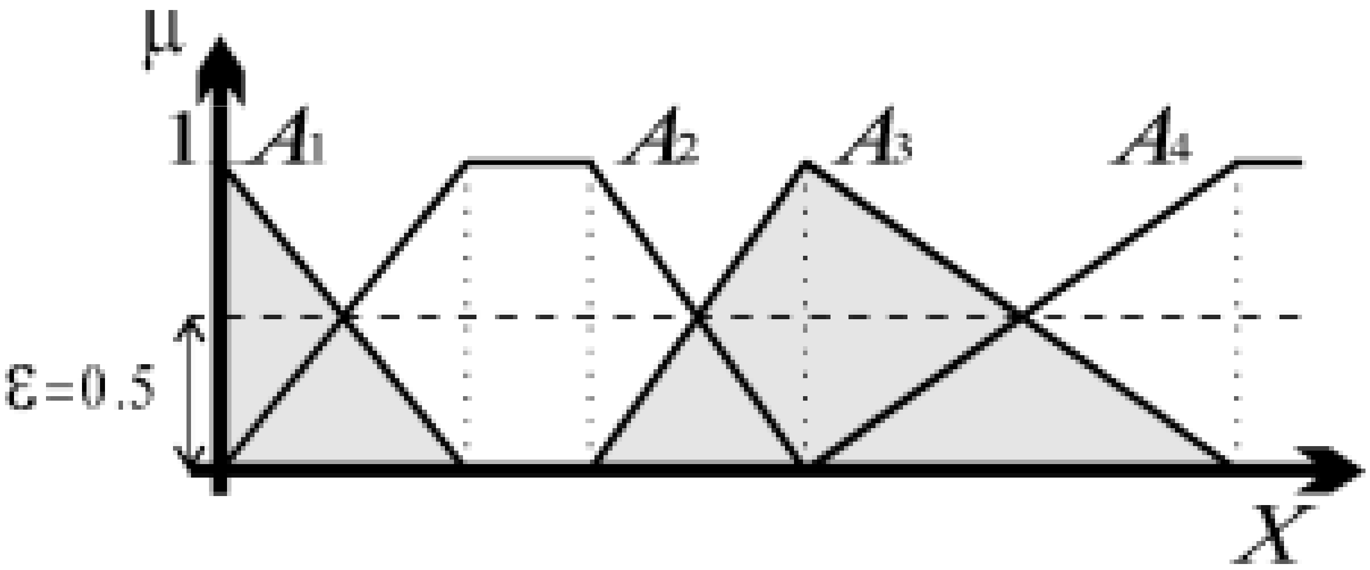


$$\forall x \in X, \exists i \in N, \mu_{A_i}(x) \geq \varepsilon$$

Fuzzy Partition

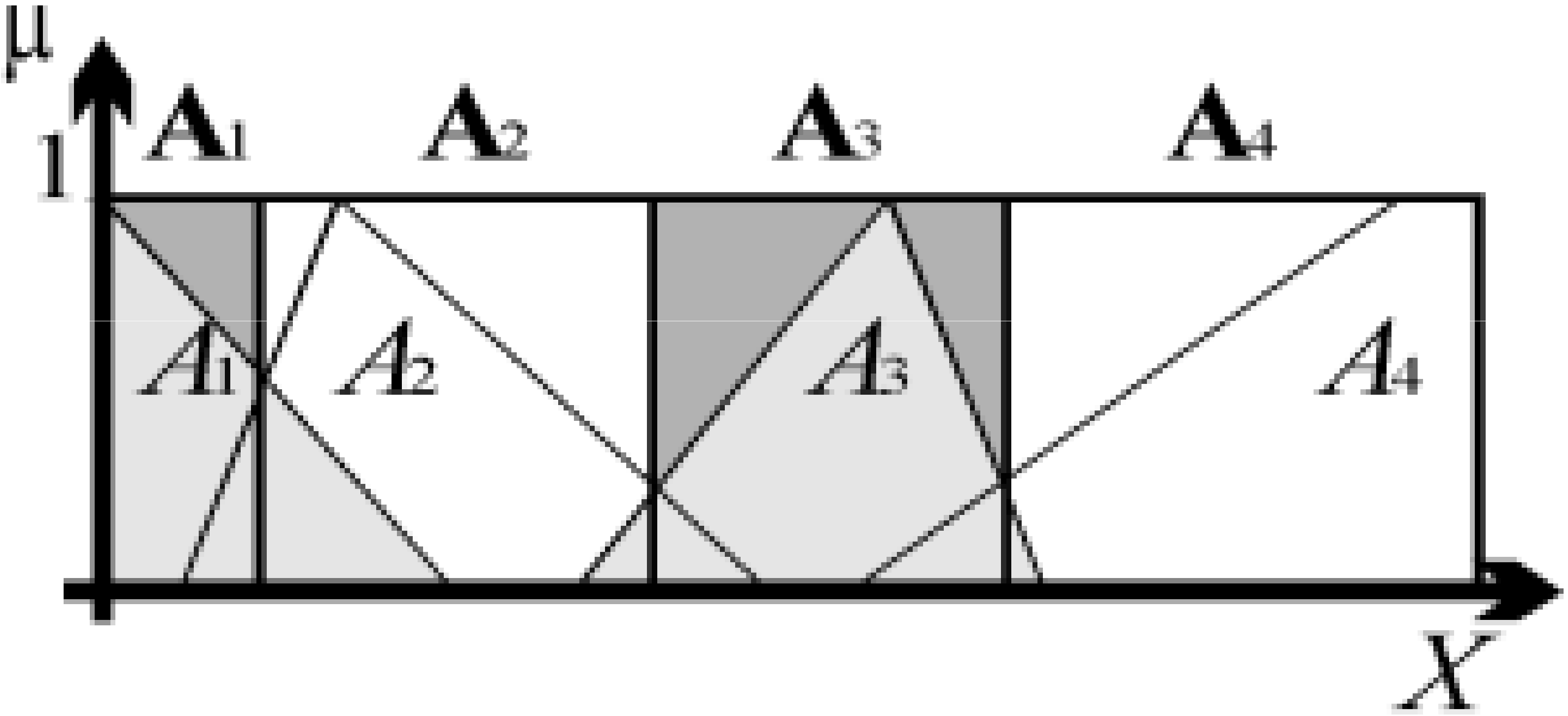
- **Ruspini-partition (0.5-cover):**

$$\sup(\mathrm{supp}(A_i(x))) = \inf(\mathrm{core}(A_{i+1}(x)))$$
$$\sup(\mathrm{core}(A_i(x))) = \inf(\mathrm{supp}(A_{i+1}(x)))$$



Boolean Partition

- **A** induced by the fuzzy partition *A***:**

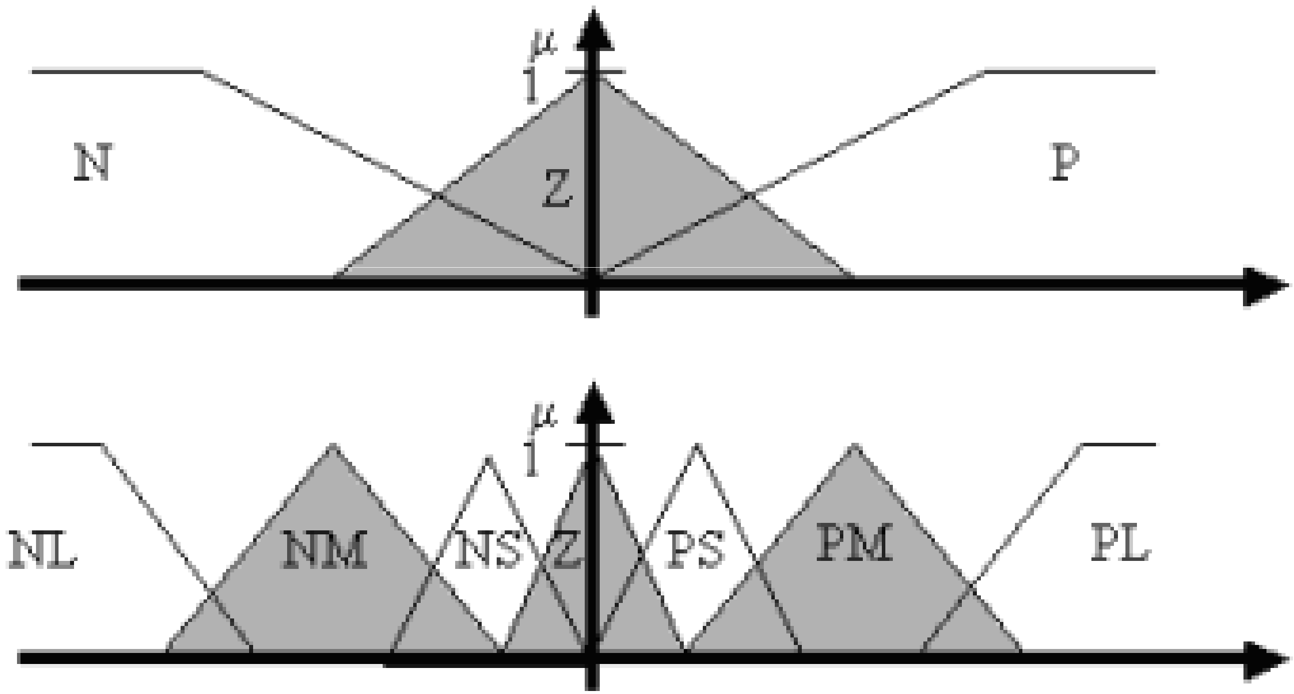


Specificity of Fuzzy Partitions

- **Fuzzy partition A' is more specific than A if all the elements of A' are more specific (e.g. in terms of their specificity measure) than the elements of A.**
- **Then, the number of elements of A' is greater than the number of linguistic terms in A.**
- **E.g. the fuzzy partition:**
  A = { *Negative*, *Zero*, *Positive*}
  is less specific than the fuzzy partition A' containing seven terms (fuzzy sets):
  A = {*Negative Large*, *Negative Middle*, *Negative Small*, *Zero*, *Positive Small*, *Positive Middle*, *Positive Large*}

Specificity of Fuzzy Partitions

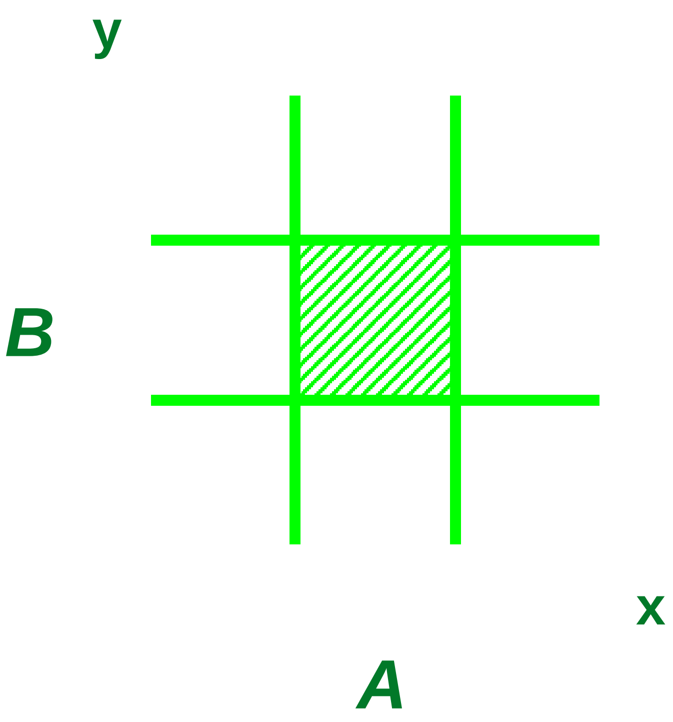
- **E.g:**



Fuzzy If-Then Rules

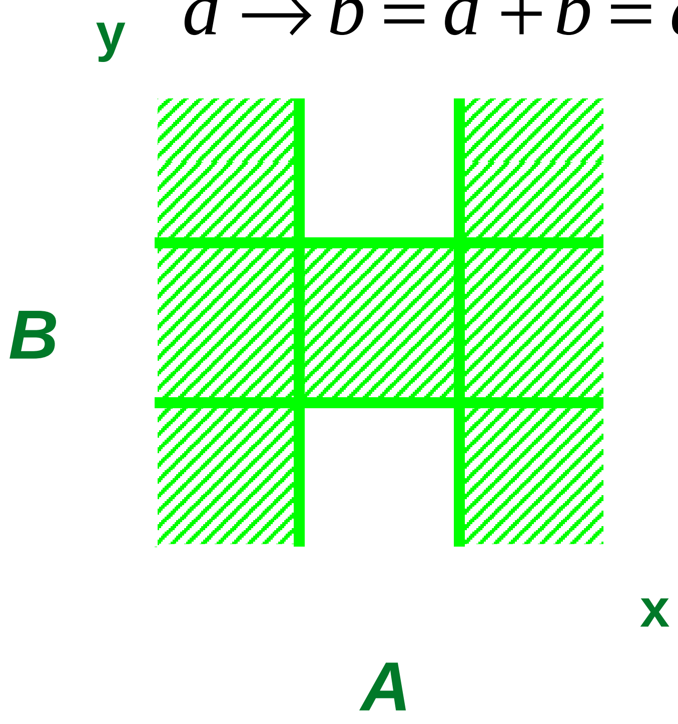
- **Two ways to interpret "If x is A then y is B":**

A coupled with B



A entails B

$$a \rightarrow b = \overline{a} + b = \overline{a \cdot \overline{b}}$$



Fuzzy If-Then Rules

- **Two ways to interpret "If x is A then y is B":**
  - **A coupled with B (Fuzzy "dot"): (*A and B*)**

$$R_i = \boldsymbol{A}_i \rightarrow \boldsymbol{B}_i$$

$$R_i = A_i \rightarrow B_i = A_i \times B_i = \int_{X \times Y} \mu_A(x) \cap \mu_B(y) \,/\, (x,y)$$

$$\mathbf{R} = \boldsymbol{A} \rightarrow \boldsymbol{B} = \bigcup^{r}_{i=1} R_i = \bigcup^{r}_{i=1} (\boldsymbol{A}_i \rightarrow \boldsymbol{B}_i)$$

  - **A entails B: (*not A or B*)**

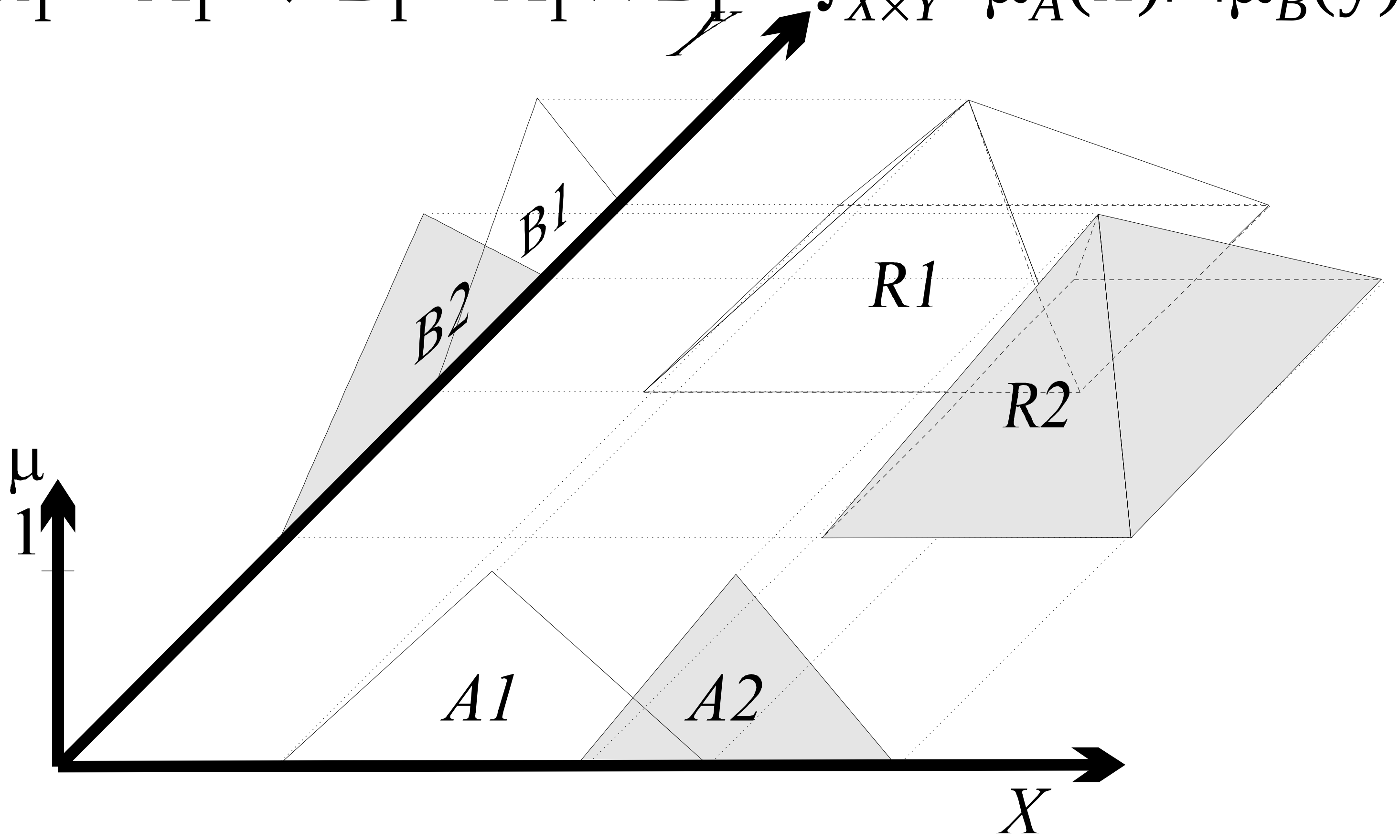
$$a \rightarrow b = \overline{a} + b = \overline{a \cdot \overline{b}}$$

    - **Material implication**
    - **Propositional calculus**
    - **Extended propositional calculus**
    - **Generalization of modus ponens**

Fuzzy If-Then Rules (Zadeh-Mamdani method)

- **A coupled with B – "Fuzzy dot":**

$$R_i = A_i \to B_i = A_i \times B_i = \int_{X\times Y} \mu_A(x) \cap \mu_B(y) \,/\, (x,y)$$



$$\mathbf{R} = \cup^{r}_{i=1} R_i = \cup^{r}_{i=1} (\boldsymbol{A}_i \to \boldsymbol{B}_i)$$

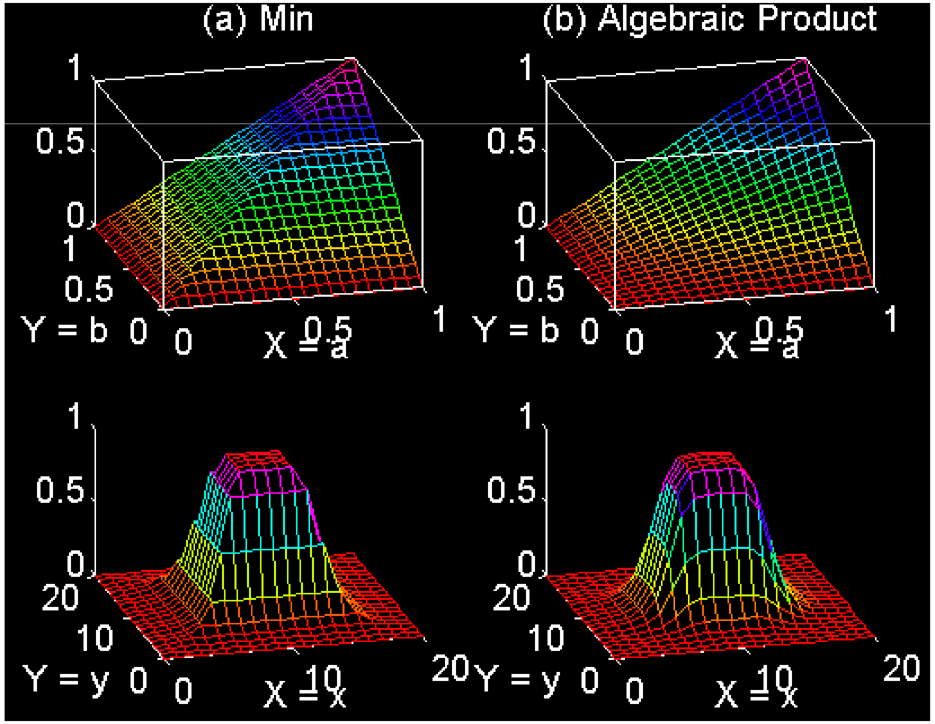
Fuzzy If-Then Rules

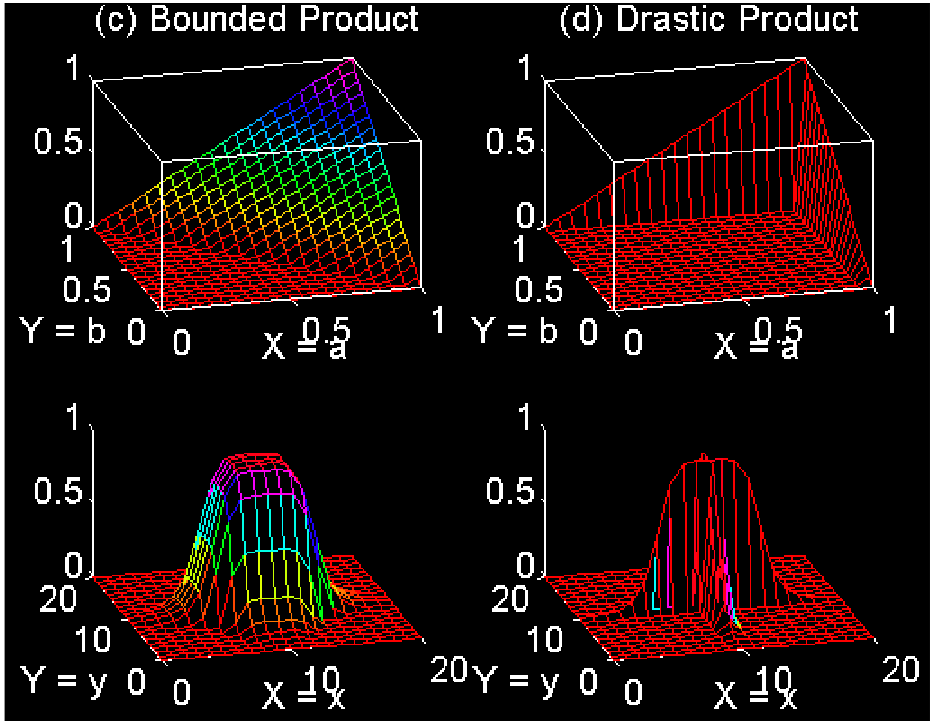
- **A coupled with B – "Fuzzy dot":**

$$R_i = \boldsymbol{A}_i \rightarrow B_i = (A_{1,i} \times A_{2,i} \times \ldots \times A_{n,i}) \rightarrow B_i = (A_{1,i} \times A_{2,i} \times \ldots \times A_{n,i}) \times B_i =$$

$$= \int_{\boldsymbol{X} \times Y} (\mu_{A_{1,i}}(x_1) \cap \mu_{A_{2,i}}(x_2) \cap \ldots \cap \mu_{A_{n,i}}(x_n)) \cap \mu_B(y) \,/\, (x_1, x_2, \ldots, x_n, y) =$$

$$= \int_{\boldsymbol{X} \times Y} \mu_{A_{1,i}}(x_1) \cap \mu_{A_{2,i}}(x_2) \cap \ldots \cap \mu_{A_{n,i}}(x_n) \cap \mu_B(y) \,/\, (x_1, x_2, \ldots, x_n, y)$$

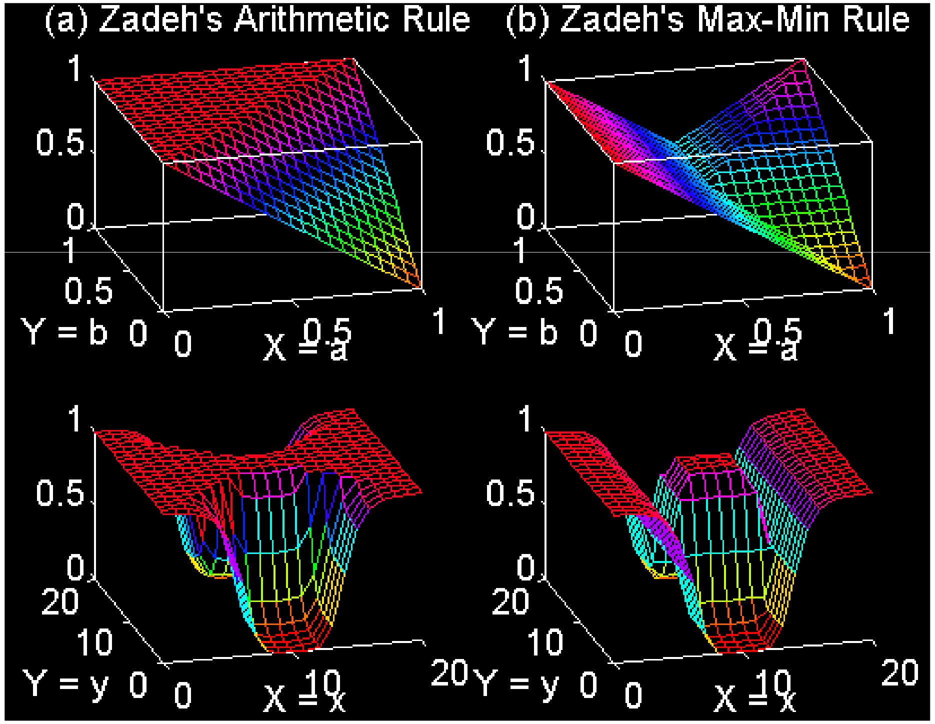


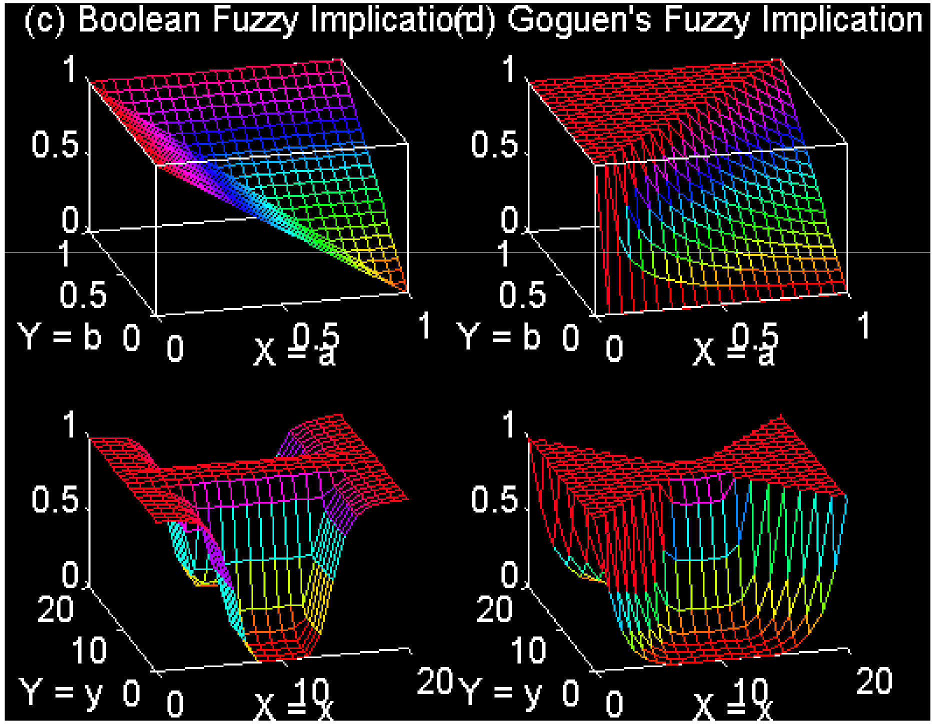


Fuzzy If-Then Rules

A entails B

$$a \rightarrow b = \overline{a} + b = \overline{a \cdot \overline{b}}$$





Max-min compositional rule of inference (Zadeh)

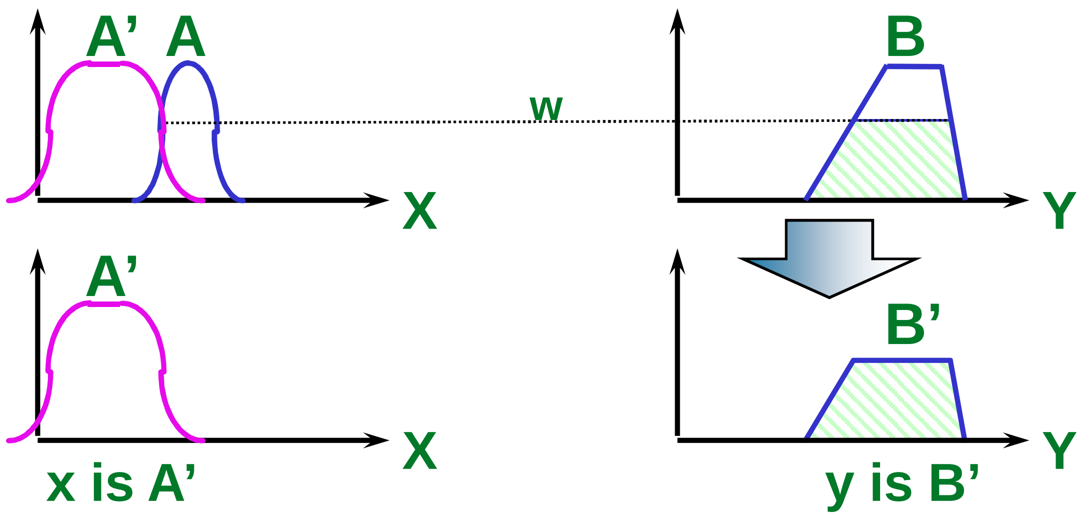
- **Single rule with single antecedent**

  Rule: If x is A then y is B

  Fact: x is A’ (Generalized modus ponens)

  Conclusion: y is B’

- **Graphic Representation:**



Max-min compositional rule of inference (Zadeh-Mamdani)

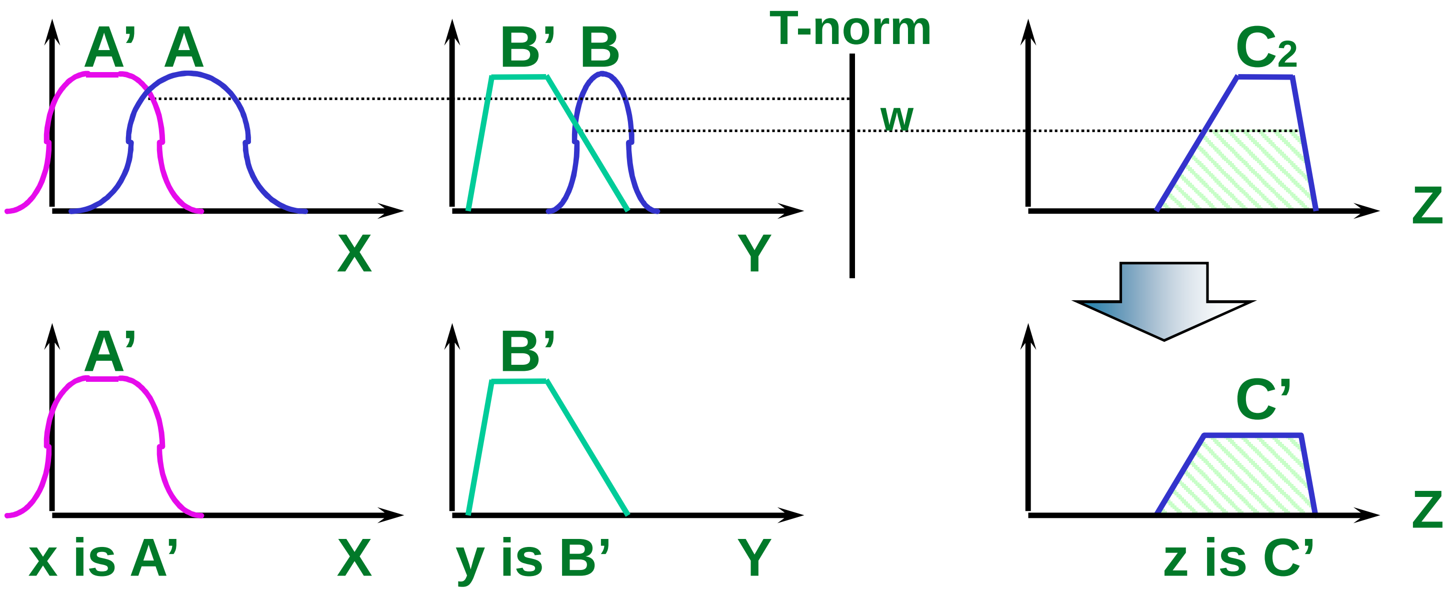
- **Single rule with multiple antecedent**

  Rule: if x is A and y is B then z is C

  Fact: x is A' and y is B'

  Conclusion: z is C'

- **Graphic Representation:**



Max-min compositional rule of inference

(Zadeh-Mamdani)

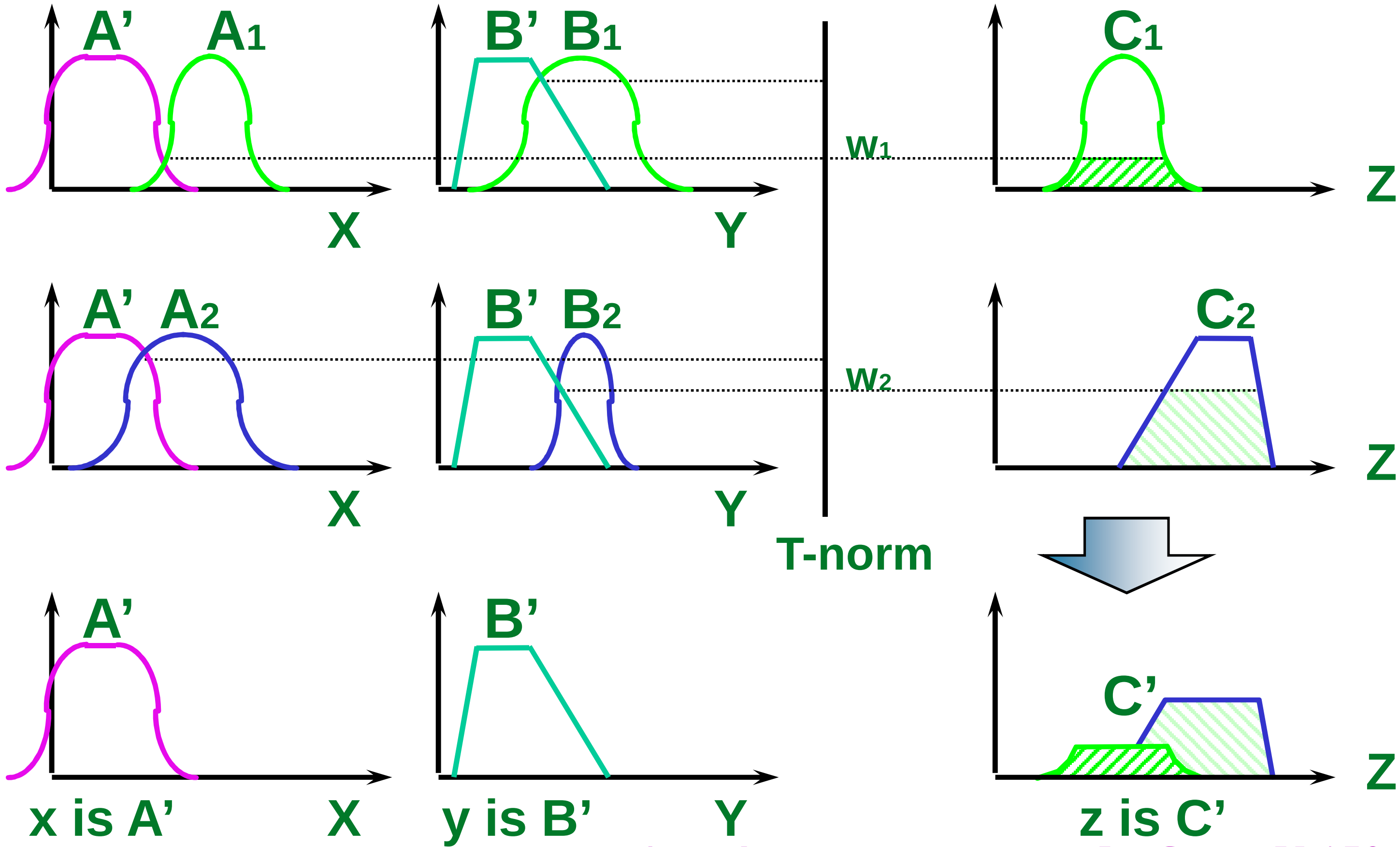
- **Multiple rules with multiple antecedent**

  Rule 1: if x is A_1 and y is B_1 then z is C_1

  Rule 2: if x is A_2 and y is B_2 then z is C_2

  Fact: x is A' and y is B'

  Conclusion: z is C'

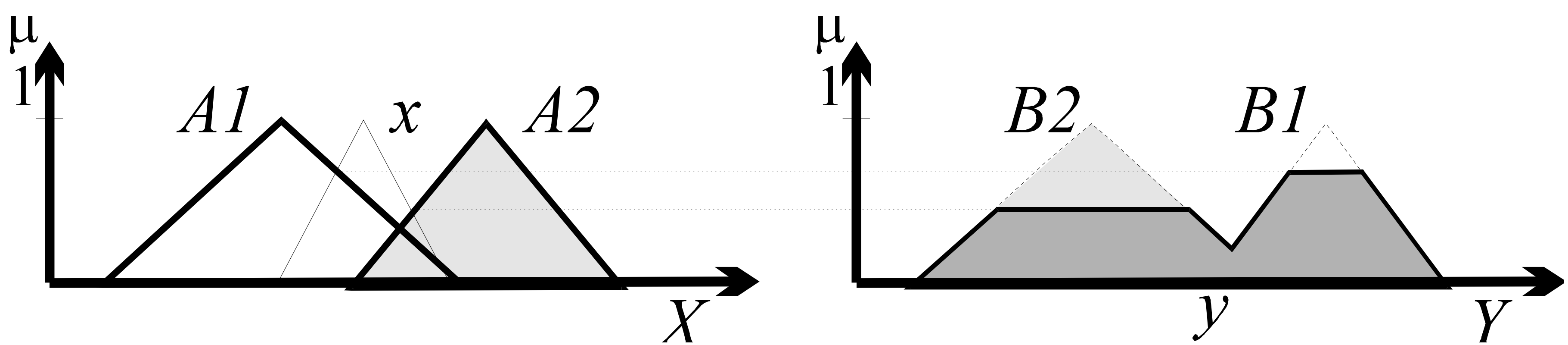


Max-min compositional rule of inference

- **Single antecedent, multiple rules – composition (Zadeh-Mamdani)**

$$y = x \circ \mathbf{R}$$

$$\mu_{x\circ\mathbf{R}}(y) = \max_{x\in X} \min[\ \mu_x(x)\ ,\ \mu_{\mathbf{R}}(x,y)\ ] \qquad \forall\ y\in Y$$



Max-min compositional rule of inference

- **Single antecedent, multiple rules – composition**

(Zadeh-Mamdani)

$y = x \circ \mathbf{R}$

$$\mu_{x\circ\mathbf{R}}(y) = \max_{x\in X} \min[\ \mu_x(x)\ ,\ \mu_{\mathbf{R}}(x,y)\ ] =$$

$$= \max_{x\in X} \min[\ \mu_x(x)\ ,\ \cup^r{}_{i=1}\ \mu_{R_i}(x,y)\ ] =$$

$$= \max_{x\in X} \min[\ \mu_x(x)\ ,\ \cup^r{}_{i=1} \min(\ \mu_{A_i}(x),\mu_{B_i}(y)\ )\ ] =$$

$$= \max_{x\in X} \cup^r{}_{i=1} \min[\ \mu_x(x)\ ,\ \min(\ \mu_{A_i}(x),\mu_{B_i}(y)\ )\ ] =$$

$$= \max_{x\in X} \cup^r{}_{i=1} \min[\ \mu_x(x)\ ,\ \mu_{A_i}(x)\ ,\ \mu_{B_i}(y)\ ] =$$

$$= \max_{x\in X} \max_{x\in X, y\in Y} (\ \min[\ \mu_x(x)\ ,\ \mu_{A_1}(x)\ ,\ \mu_{B_1}(y)\ ]\ ,$$
$$\min[\ \mu_x(x)\ ,\ \mu_{A_2}(x)\ ,\ \mu_{B_2}(y)\ ]\ ,\ \ldots\ ,$$
$$\min[\ \mu_x(x)\ ,\ \mu_{A_r}(x)\ ,\ \mu_{B_r}(y)\ ]\ ) =$$

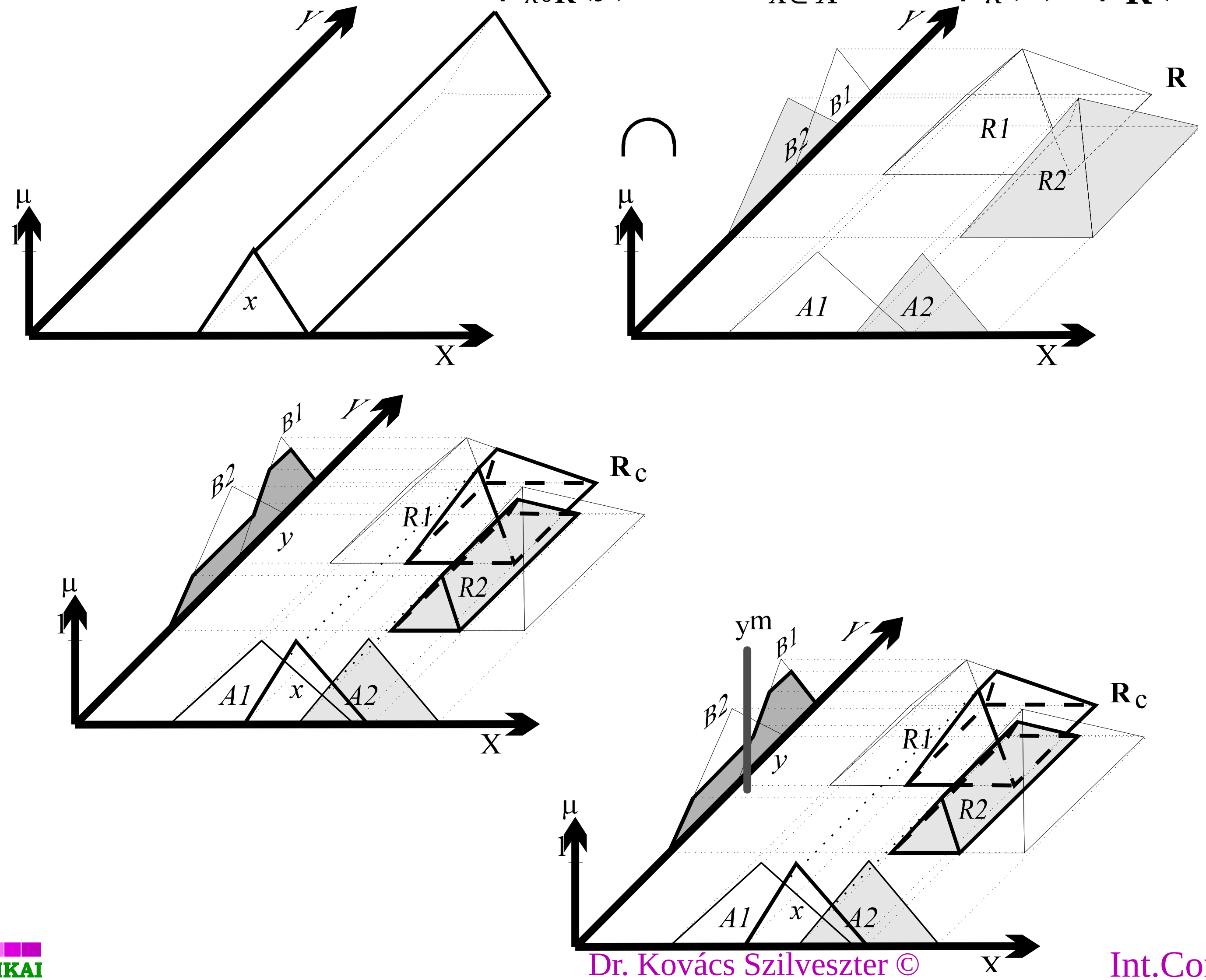
$$= \cup^r{}_{i=1} \max_{x\in X} \min[\ \mu_x(x)\ ,\ \mu_{A_i}(x)\ ,\ \mu_{B_i}(y)\ ] =$$

$$= \cup^r{}_{i=1}\ \mu_{x\circ R_i}(y) \qquad \forall\ y\in Y$$

Max-min compositional rule of inference (Zadeh-Mamdani)

- **Single antecedent** $y = x \circ \mathbf{R}$

$\mu_{x\circ\mathbf{R}}(y) = \max_{x\in X} \min[\, \mu_x(x)\,,\, \mu_{\mathbf{R}}(x,y)\,] \quad \forall\, y\in Y$



Max-min compositional rule of inference (Zadeh-Mamdani)

- **Multiple antecedents – composition**

$y = (x_1 , x_2 ,..., x_n) \circ \mathbf{R}$

$$\mu_{(x1,x2,...,xn)\circ\mathbf{R}}(y) = \max_{x_1,x_2,...,x_n} \min[\ \mu_{x_1}(x_1),\mu_{x_2}(x_2),...,\mu_{x_n}(x_n)\ ,\ \mu_{\mathbf{R}}(x,y)\ ] =$$

$$= \max_{x_1,x_2,...,x_n} \min[\ \mu_{x_1}(x_1),\mu_{x_2}(x_2),...,\mu_{x_n}(x_n)\ ,\ \cup^{r}_{i=1}\ \mu_{R_i}(x_1,x_2,..,x_n,y)\ ] =$$

$$= \max_{x_1,x_2,...,x_n} \min[\ \mu_{x_1}(x_1),\mu_{x_2}(x_2),...,\mu_{x_n}(x_n)\ ,$$
$$\cup^{r}_{i=1} \min(\ \mu_{A_{1,i}}(x_1),\mu_{A_{2,i}}(x_2),...,\mu_{A_{n,i}}(x_n),\mu_{B_i}(y)\ )\ ] =$$

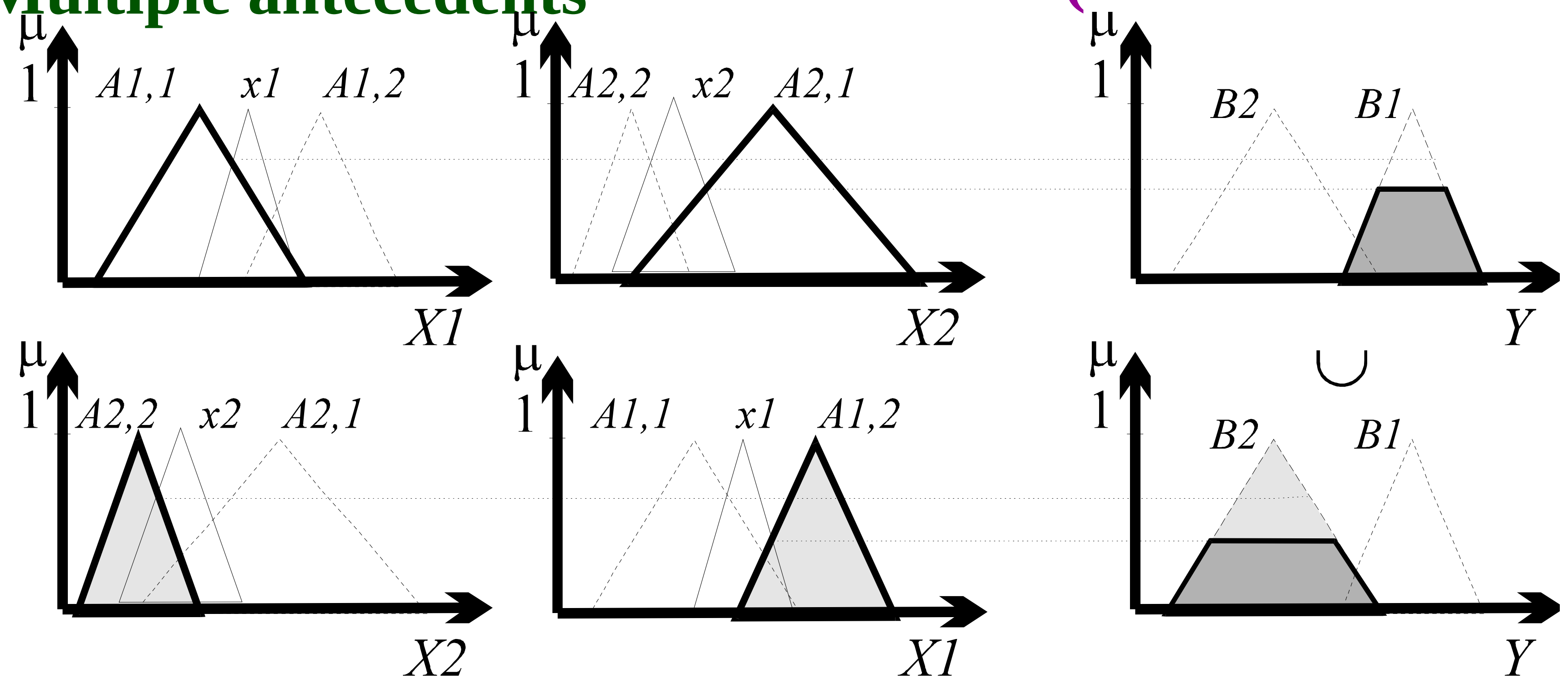
$$= \max_{x_1,x_2,...,x_n} \cup^{r}_{i=1} \min[\ \mu_{x_1}(x_1),\mu_{x_2}(x_2),...,\mu_{x_n}(x_n)\ ,$$
$$\mu_{A_{1,i}}(x_1),\mu_{A_{2,i}}(x_2),...,\mu_{A_{n,i}}(x_n),\mu_{B_i}(y)\ )\ ] =$$

$$= \cup^{r}_{i=1} \max_{x_1,x_2,...,x_n} \min[\ \mu_{x_1}(x_1),\mu_{x_2}(x_2),...,\mu_{x_n}(x_n)\ ,$$
$$\mu_{A_{1,i}}(x_1),\mu_{A_{2,i}}(x_2),...,\mu_{A_{n,i}}(x_n),\mu_{B_i}(y)\ )\ ] =$$

$$= \cup^{r}_{i=1}\ \mu_{(x1,x2,...,xn)\circ R_i}(y) \qquad \forall\ y \in Y$$

Max-min compositional rule of inference (Zadeh-Mamdani)

- **Multiple antecedents**

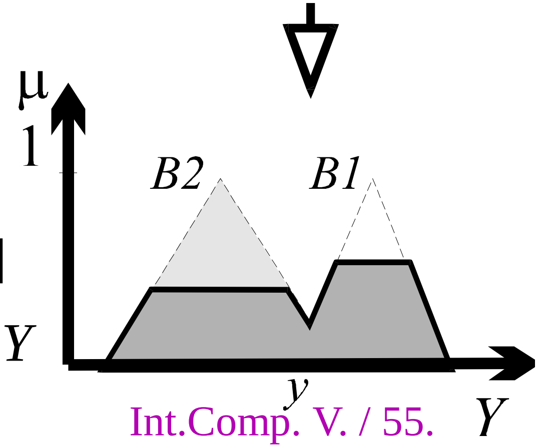


$y = (x_1 , x_2 ,..., x_n) \circ \mathbf{R}$

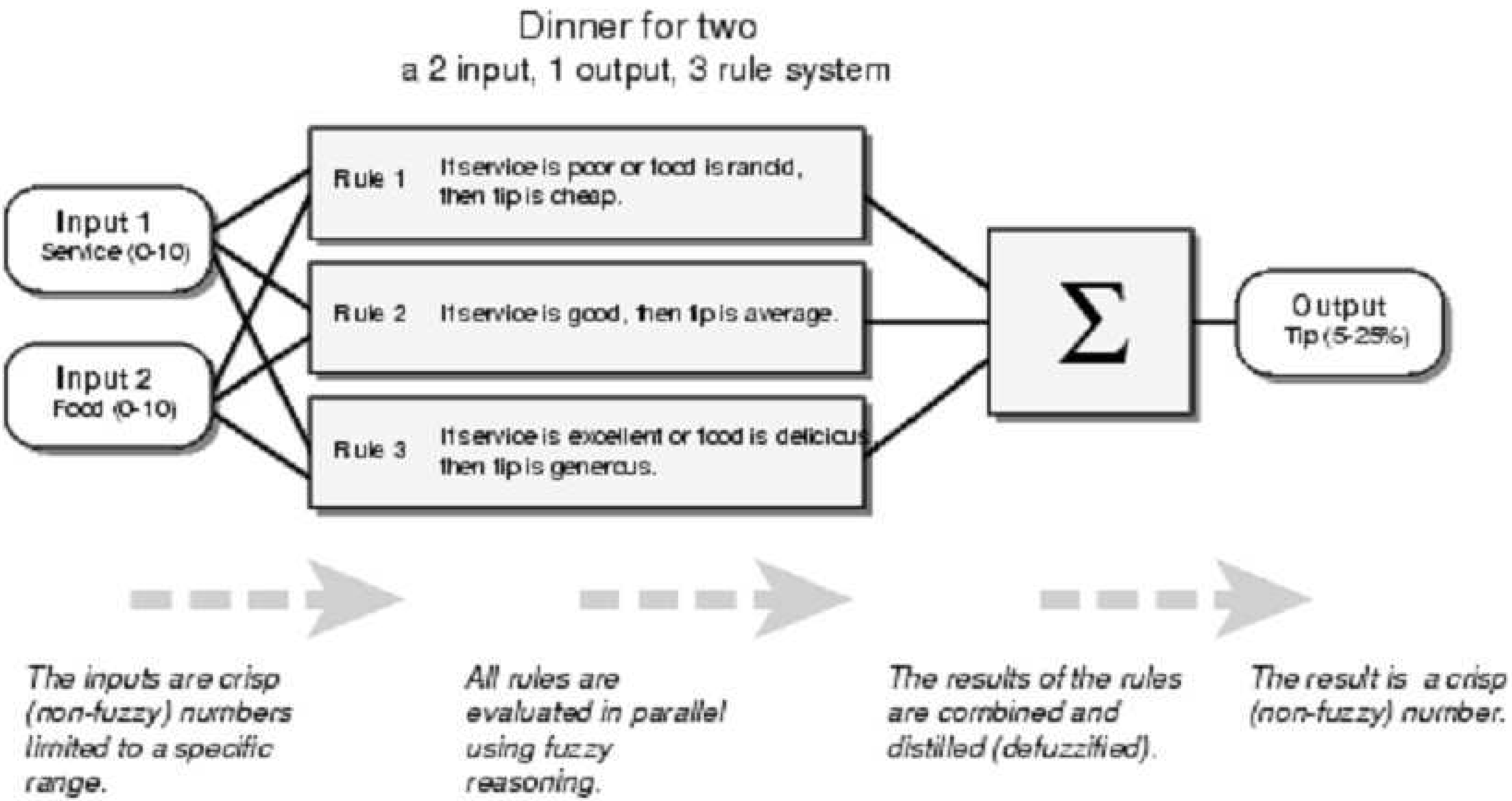
$\mu_{(x1,x2,...,xn)\circ \mathbf{R}}(y) =$

$= \max_{x_1,x_2,...,x_n} \min[\mu_{x_1}(x_1),\mu_{x_2}(x_2),...,\mu_{x_n}(x_n) , \mu_{\mathbf{R}}(x,y)]$

$\forall\ y \in Y$



Fuzzy inference, parallel implementation



Defuzzification

- **The last step in the fuzzy inference process is defuzzification.**
- **Fuzziness helps us to evaluate the rules, but the final output of a fuzzy system has to be a crisp number.**
- **The input for the defuzzification process is the aggregate output fuzzy set and the output is a single (crisp) number.**

Defuzzification requirements

Intuition:

- **A crisp value should represent the fuzzy set from an intuitive point of view (e.g., max. membership grade)**

Computational Burden:

- **simple (real-time constraints)**

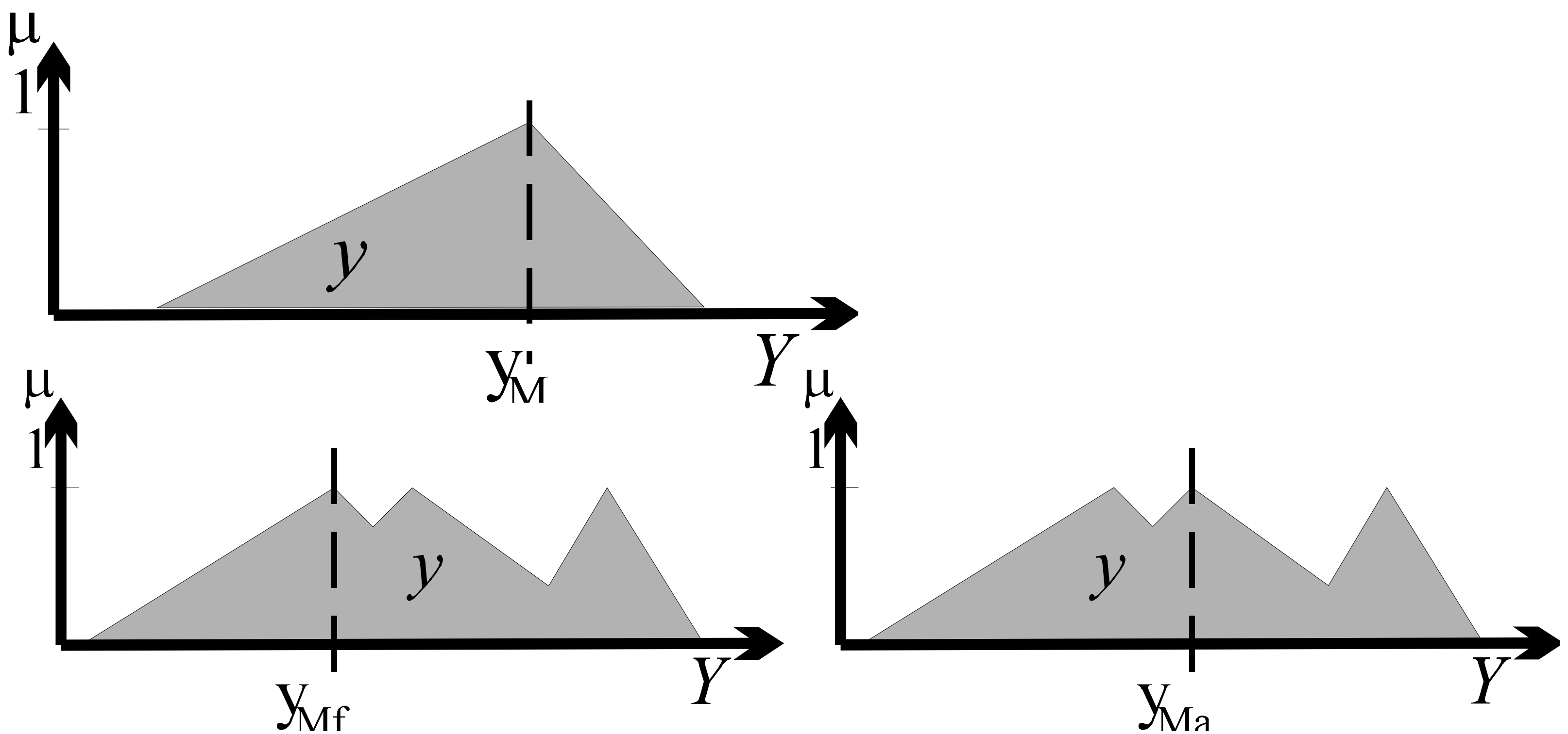
Continuity:

- **small changes in fuzzy sets should not result in large changes of the consequent**

Defuzzification

- **The Max Criterion Method**

$$y_M : \quad y_M \in Y, \qquad \mu_y(y_M) = \max_{y \in Y} \mu_y(y)$$

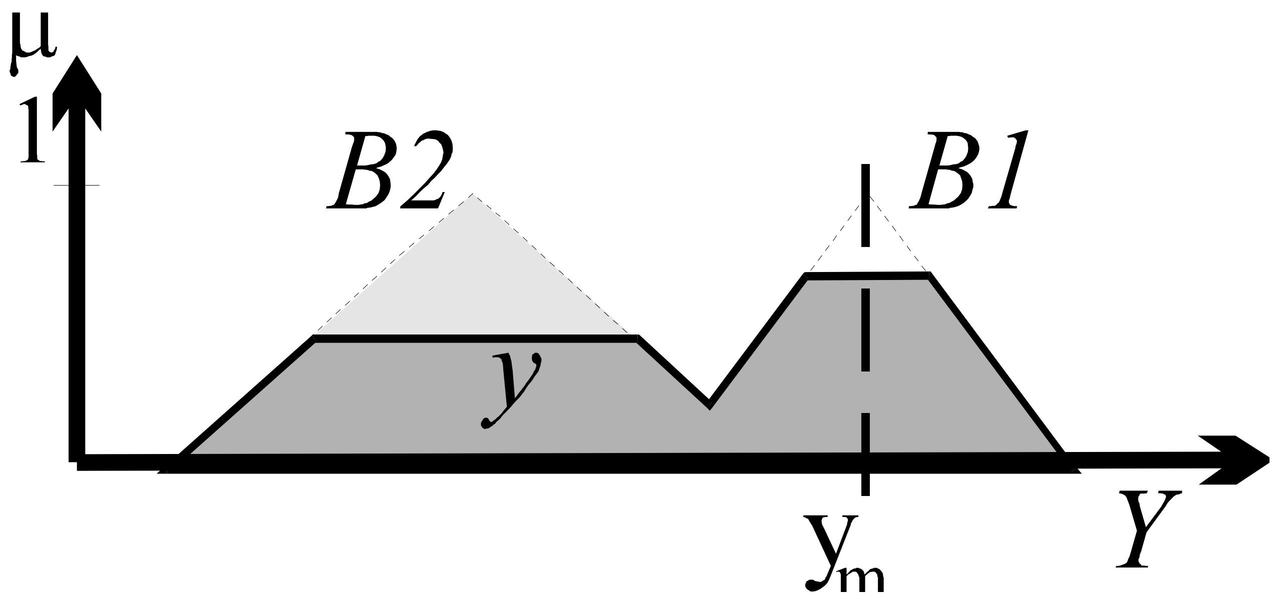


Defuzzification

- **The Mean of Maxima Method (MOM)**

$$y_M : \{ \ y^i_M \in Y \mid \mu_y(y^i_M) = \max_{y\in Y} \mu_y(y) \ \}$$

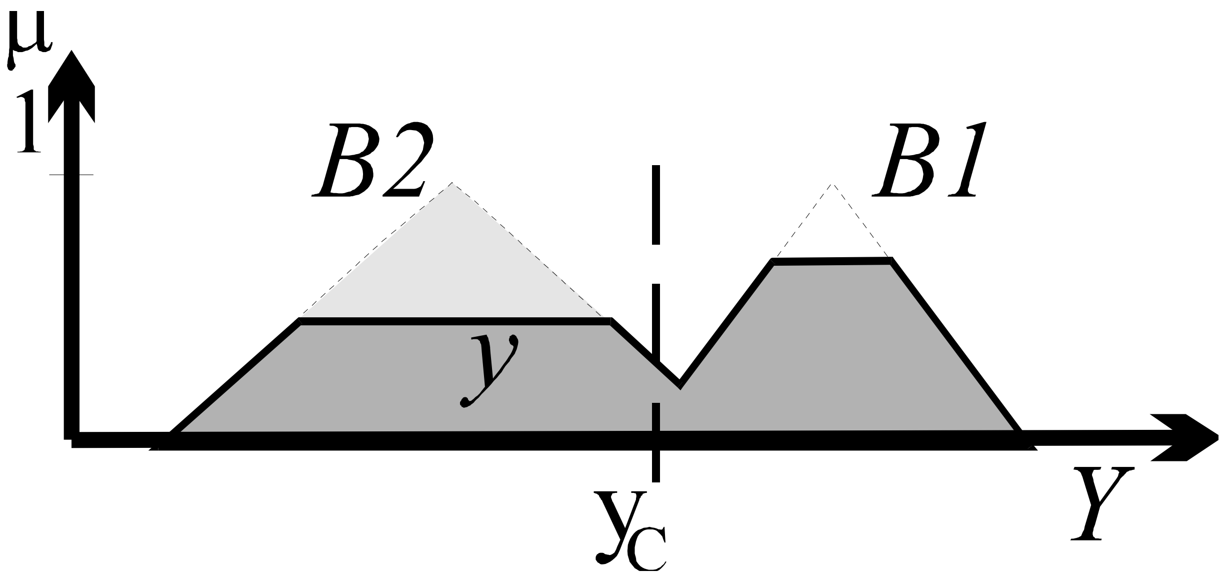
$$y_m = \sum^n_{i=1}(\ y^i_M / n \ )$$



Defuzzification

- **The Center of Gravity Method (COG)**

$$y_c = \sum_{y \in Y}(\ \mu_y(y) \cdot y\ )\ /\ \sum_{y \in Y}\ \mu_y(y)$$

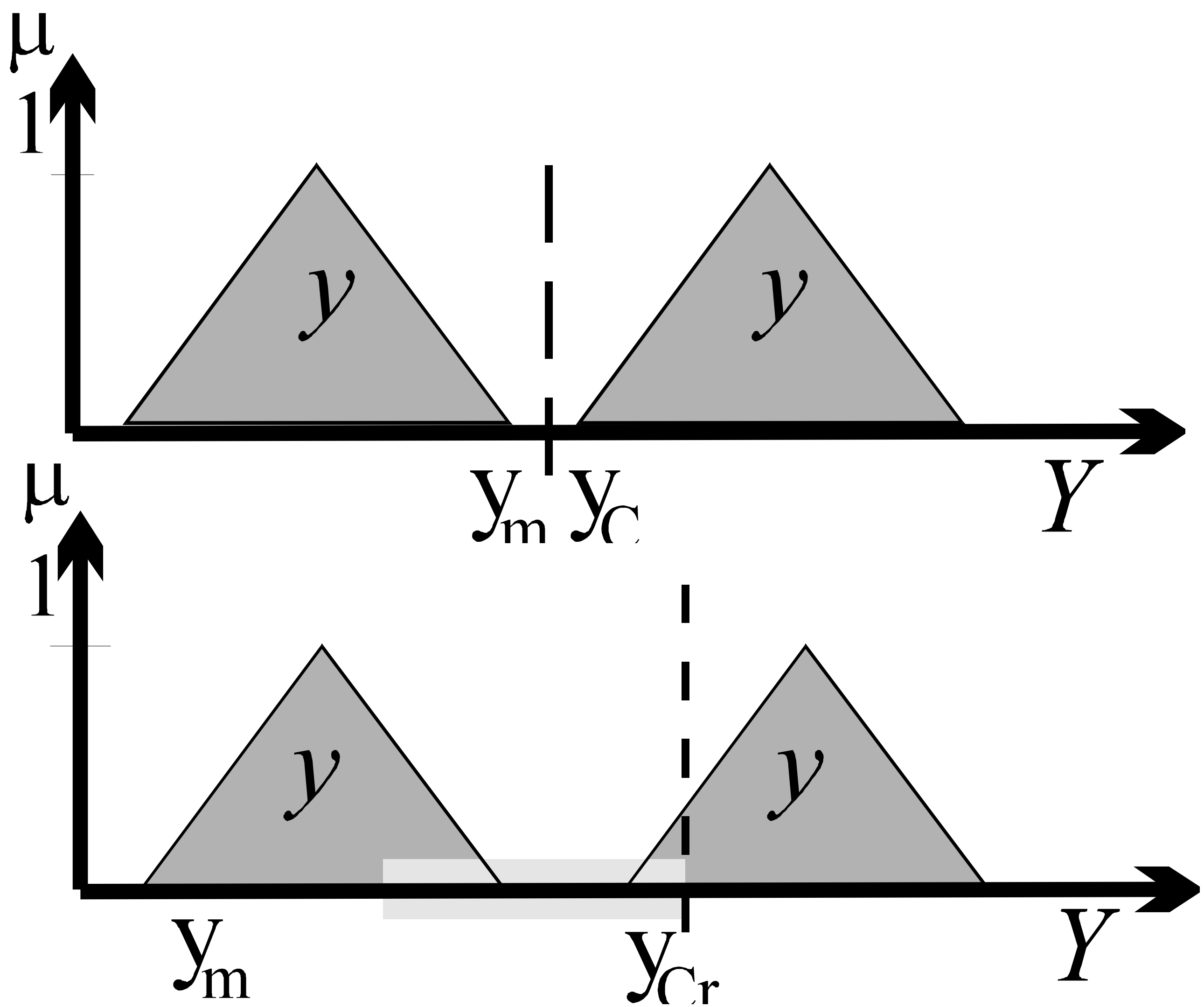


$$y_c = \frac{\int_Y \mu_y(y) \cdot y \cdot dy}{\int_Y \mu_y(y) \cdot dy}$$

+ intuitive
+ smooth
- computational burden

Defuzzification

- **Defuzzification with additional restrictions**



Defuzzification

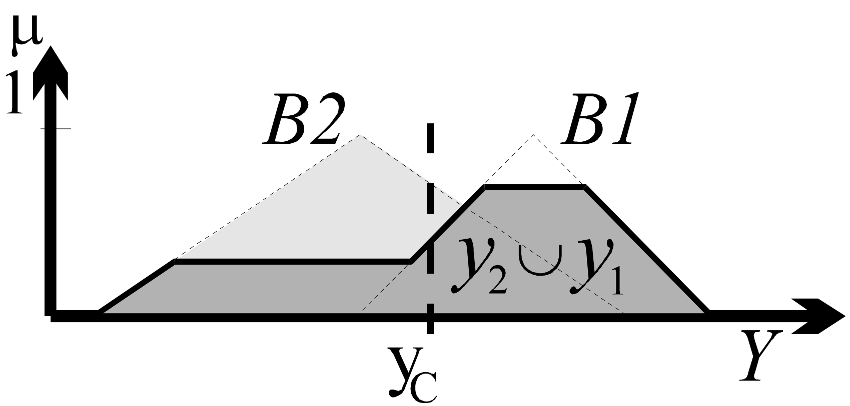
- **The Center of Sums Method**

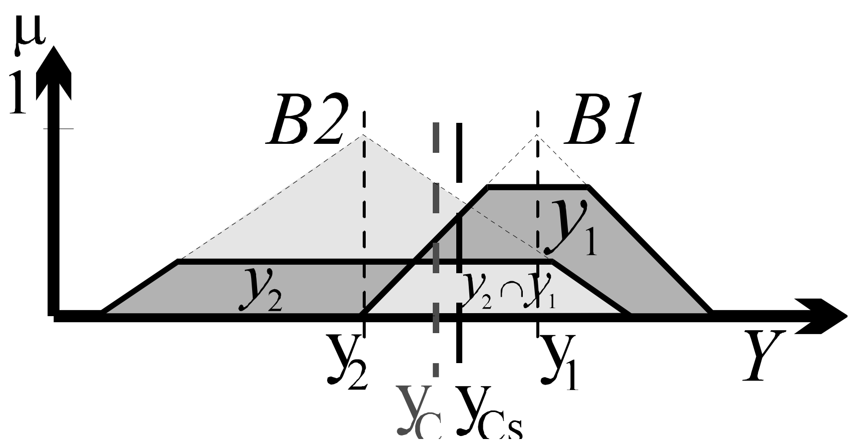
$$y = y_1 \cup y_2 \cup \ldots \cup y_n$$

$$w_i = \sum_{y \in Y} \mu_{y_i}(y)$$

$$y_i = \sum_{y \in Y} \left(\mu_{y_i}(y) \cdot y \right) / \sum_{y \in Y} \mu_{y_i}(y)$$

$$y_{cs} = \sum_{i \in [1,n]} \left(w_i \cdot y_i \right) / \sum_{i \in [1,n]} w_i$$

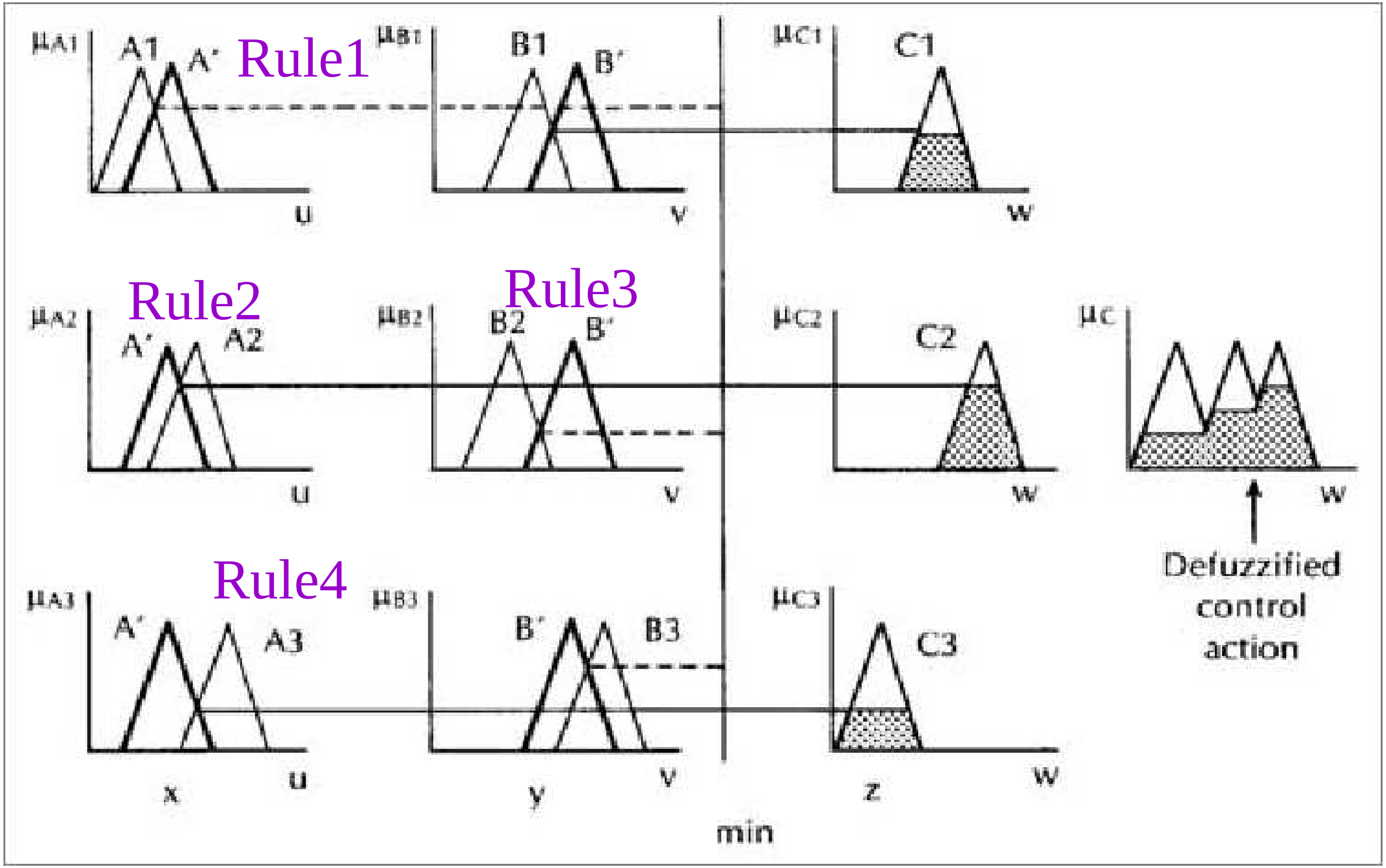




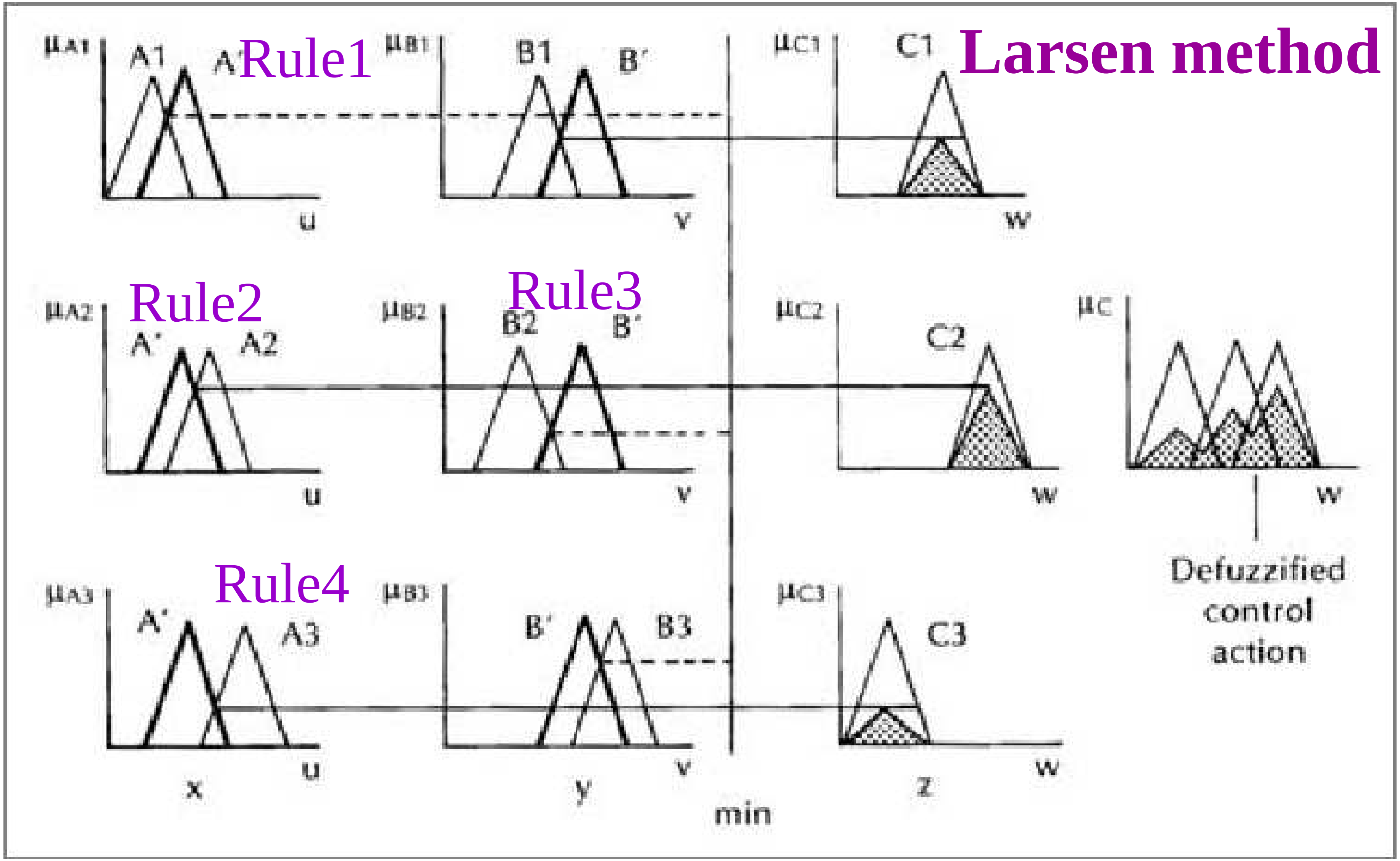
Compositional rule of inference - example

- Rule1:
  IF x is A1 AND y is B1 THEN z is C1
- Rule2:
  IF x is A2 THEN z is C2
- Rule3:
  IF y is B2 THEN z is C2
- Rule4:
  IF x is A3 AND y is B3 THEN z is C3
- Observations:
  x = A'
  y = B'

Compositional rule of inference: Max-min



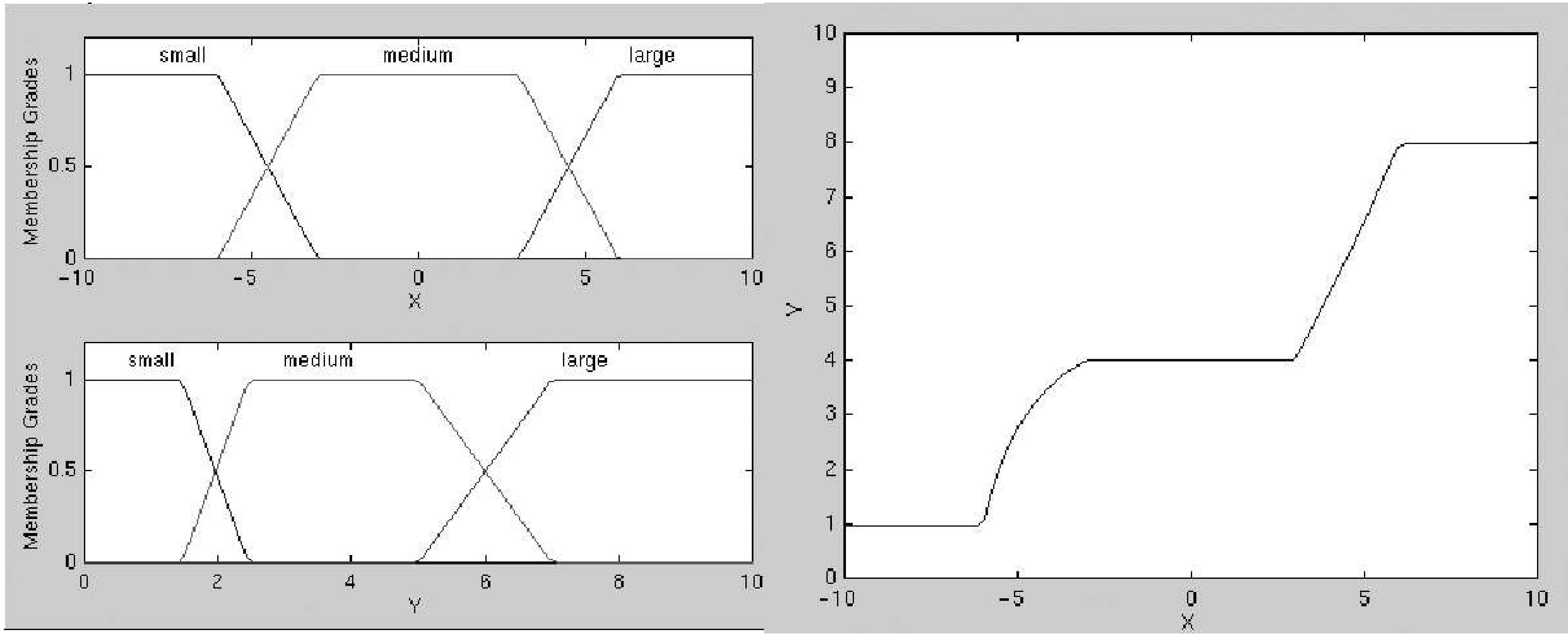
Compositional rule of inference: Max-product



Compositional rule of inference - example

SISO: max-min composition center of gravity defuzzification

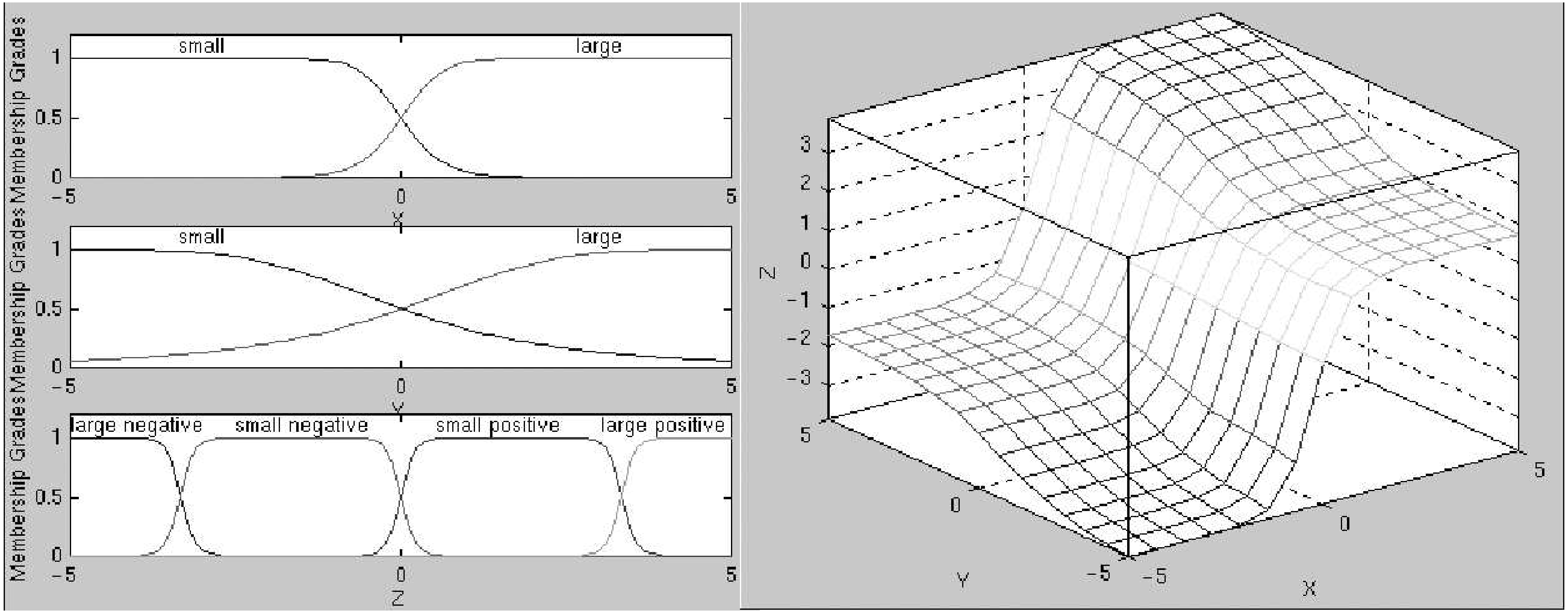
- **If X is small then Y is small**
- **If X is medium then Y is medium**
- **If X is large then Y is large**



Compositional rule of inference - example

MISO: max-min composition center of gravity defuzzification

- **If X is small and Y is small then Z is negative large**
- **If X is small and Y is large the Z is negative small**
- **If X is large and Y is small the Z is positive small**
- **If X is large and Y is large then Z is positive large**



Sugeno fuzzy inference

- **Sugeno-style (Michio Sugeno) fuzzy inference is partly similar to the Mamdani method. (Does not follow compositional rule of inference.)**
- **Sugeno changed the rule consequent. Instead of a fuzzy set, he used a mathematical function of the input variable.**
- **The format of the Sugeno-style fuzzy rule is**

$$\textbf{IF}\ \ x \textbf{ is } A\ \ \textbf{AND}\ \ y \textbf{ is } B\ \ \textbf{THEN}\ \ z \textbf{ is } f(x, y)$$

where x, y **and** z **are linguistic variables;** A **and** B **are fuzzy sets on universe of discourses** X **and** Y**, respectively; and** $f(x, y)$ **is a mathematical function.**

Sugeno fuzzy inference

- **The format of the ith rule is**

IF $x_1 = A_{1,i}$ **AND** $x_2 = A_{2,i}$ **AND** … **AND** $x_n = A_{n,i}$
THEN $y_i = f_i(x_1, x_2, \dots, x_n)$

- **The conclusion:**

$$y = \frac{\sum_{i=1}^{r} w_i \cdot y_i}{\sum_{i=1}^{r} w_i} = \frac{\sum_{i=1}^{r} w_i \cdot f\left(x_1, x_2, \dots, x_n\right)_i}{\sum_{i=1}^{r} w_i}$$

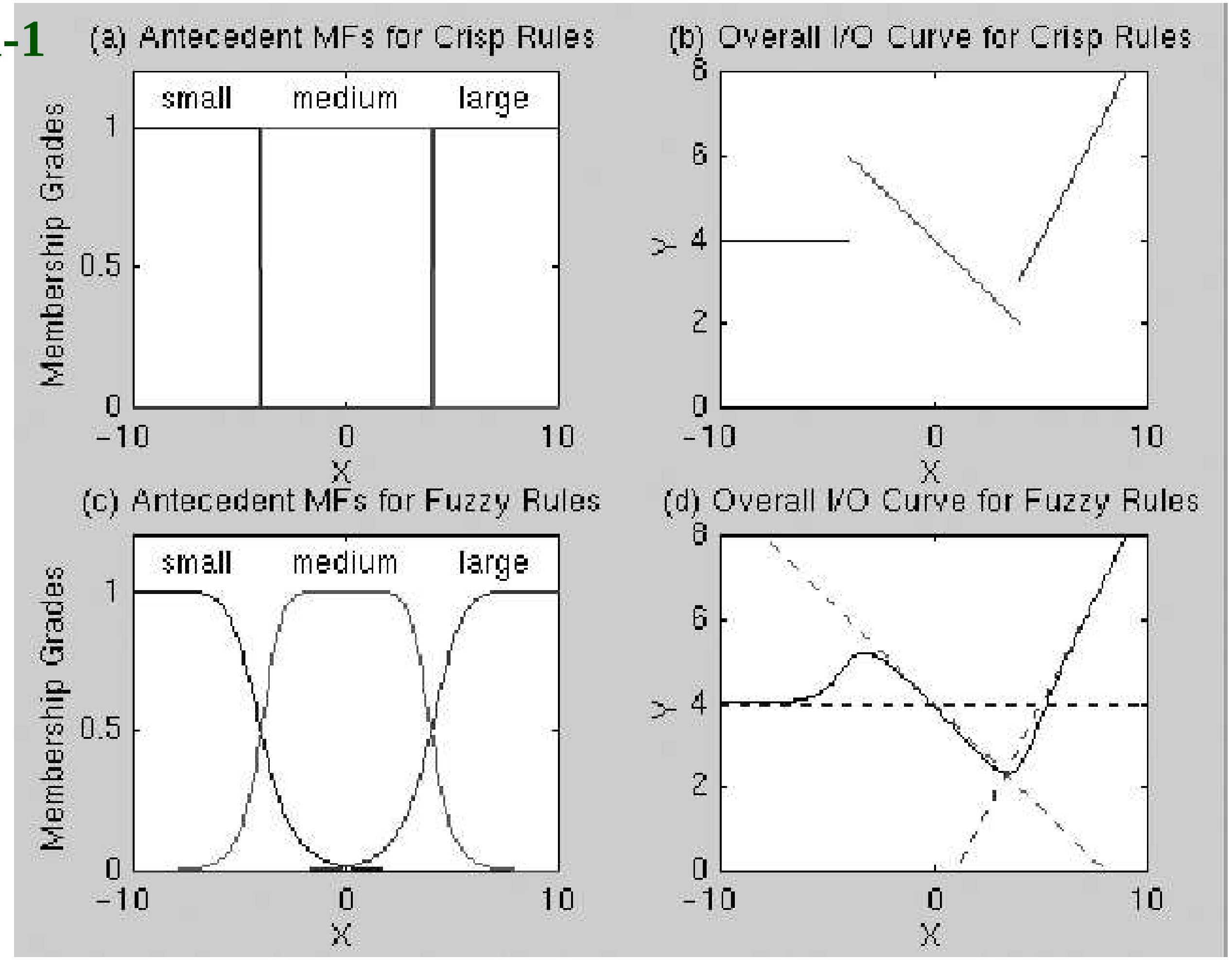
where

$$w_{j,i} = \max_j \left\{ \min\left(\mu_x\left(x_j\right), \mu_{A_{j,i}}\left(x_j\right) \right) \right\}$$

$$w_i = \min\left(w_{1,i}, w_{2,i}, \dots, w_{n,i} \right)$$

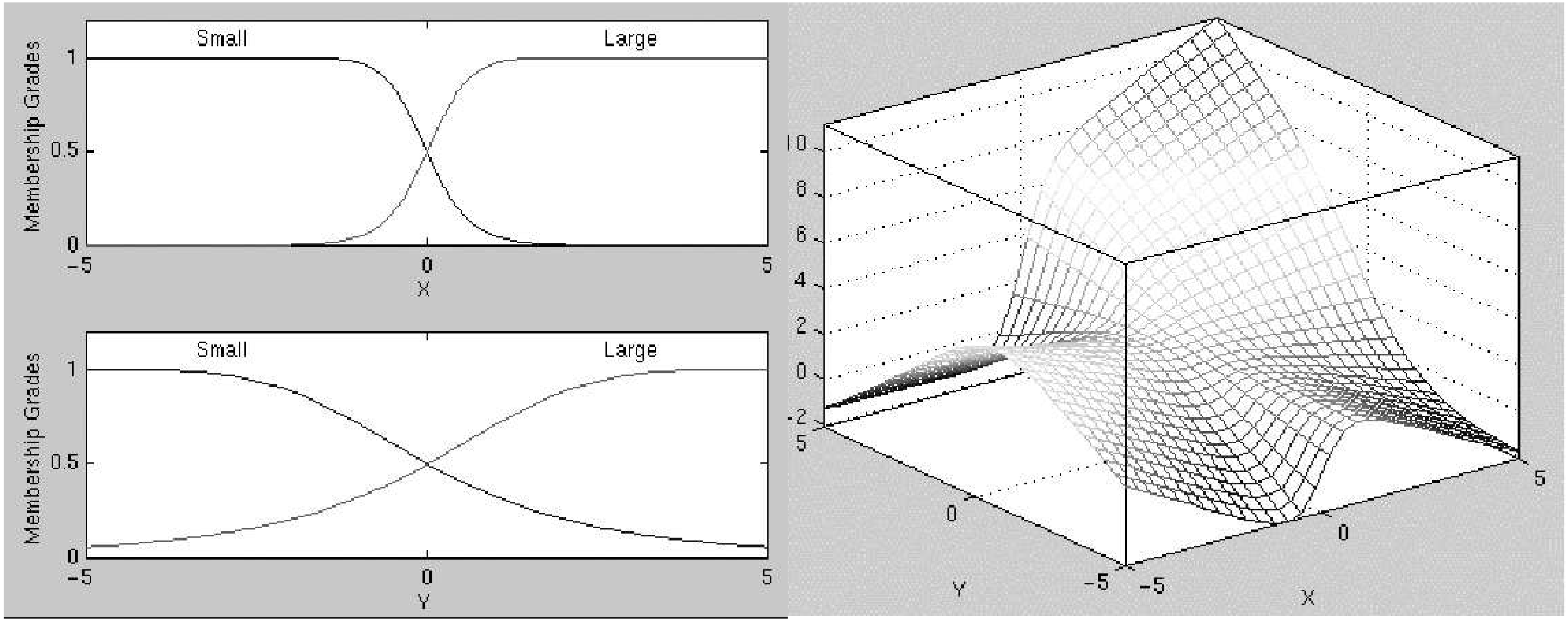
Sugeno fuzzy inference - SISO

- **Combines fuzzy sets in antecedents with crisp function in output:**
- **IF x is small THEN Y=4**
- **IF X is medium THEN Y=-0.5X+4**
- **IF X is large THEN Y=X-1**



Sugeno fuzzy inference - MISO

- **IF X is small AND Y is small THEN z=-x+y+1**
- **IF X is small AND Y is large THEN z=-y+3**
- **IF X is large and Y is small THEN z=-x+3**
- **IF X is large and Y is large THEN z=x+y+2**



Zero order Sugeno - Takagi Sugeno fuzzy inference

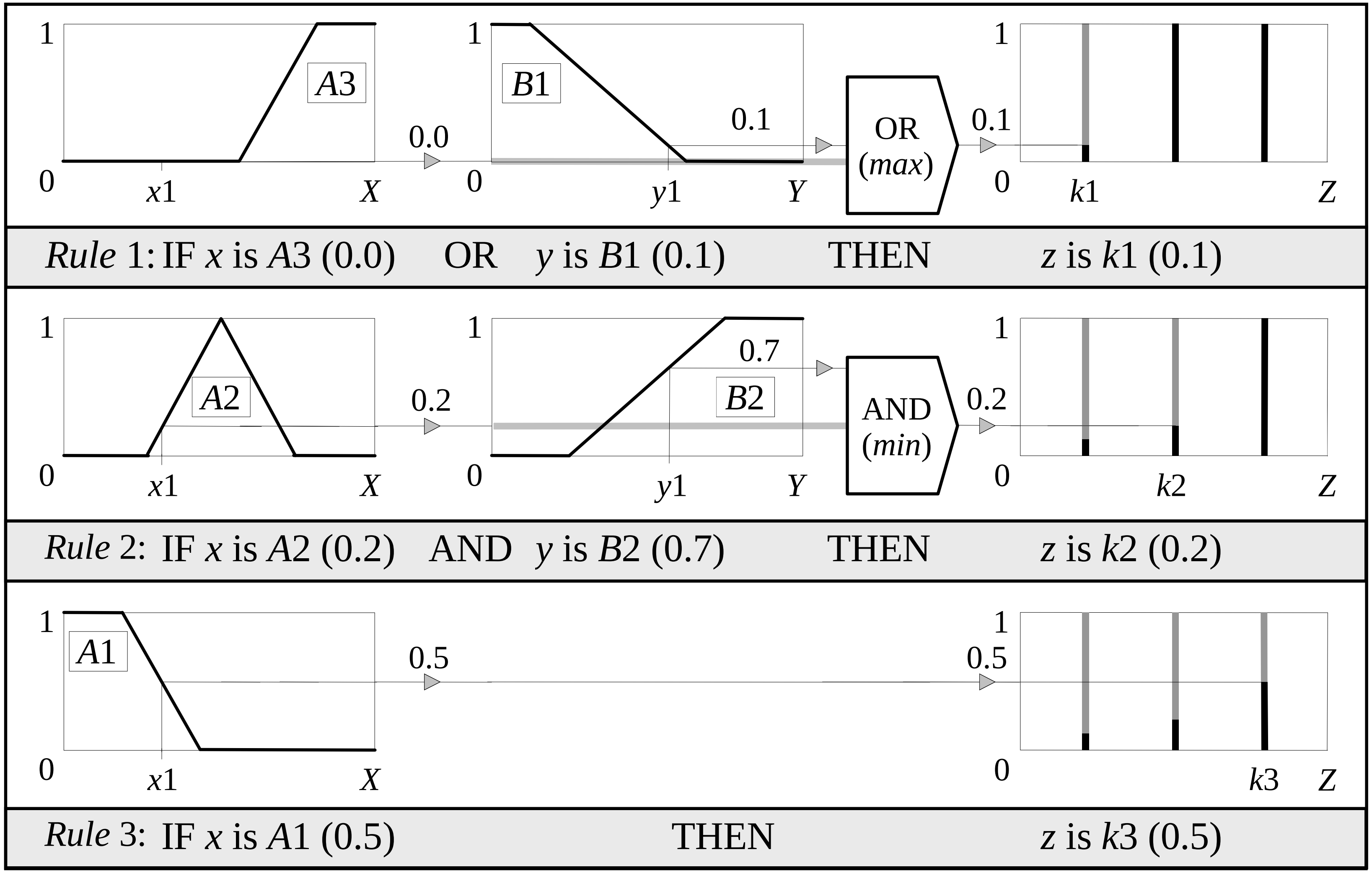
- **The most commonly used zero-order Sugeno fuzzy model applies fuzzy rules in the following form:**

  IF x **is** A
  AND y **is** B
  THEN z **is** k

  where k **is a constant.**

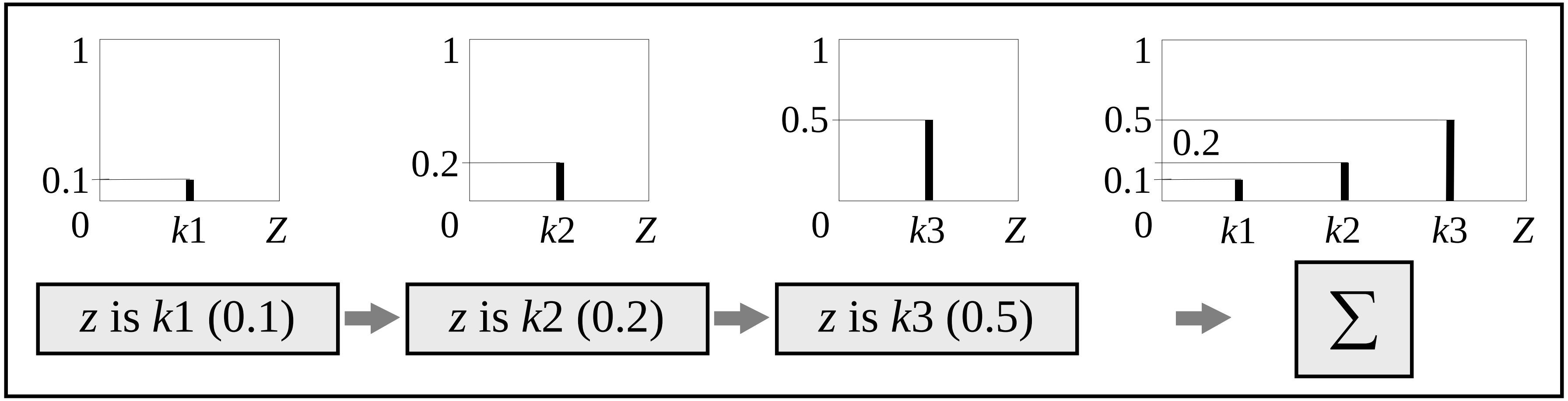
- **In this case, the output of each fuzzy rule is constant.**
- **All consequent membership functions are represented by singleton spikes.**

Zero order Sugeno fuzzy inference - example



Zero order Sugeno fuzzy inference - example

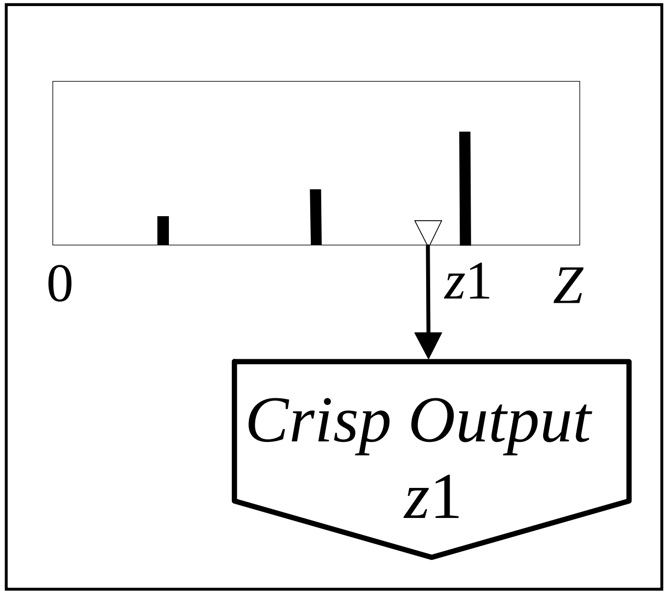
- **Aggregation of the rule outputs**



Zero order Sugeno fuzzy inference - example

- **Aggregation of the rule outputs by weighted average (WA)**
- **No need for defuzzification**

$$WA = \frac{\mu(k1)\times k1 + \mu(k2)\times k2 + \mu(k3)\times k3}{\mu(k1)+\mu(k2)+\mu(k3)} = \frac{0.1\times 20 + 0.2\times 50 + 0.5\times 80}{0.1+0.2+0.5} = 65$$



Design aspects of fuzzy inference system

Consistency:

- **no rules with the same antecedents but different consequents**

Completeness:

- **for any observation there is at least one rule firing**

Advantages of using fuzzy logic

- **Conceptually easy to understand**
  The mathematical concepts behind fuzzy reasoning are very simple.
- **Flexible**
  Fuzzy rules can be easily modified and added with starting from scratch.
- **Tolerant of imprecise data**
- **Can model nonlinear functions of arbitrary complexity.**
- **Can create a fuzzy system to match any set of input-output data (universal approximator).**
- **Can be built on top of the experience of experts.**
- **Is close to natural language.**

Disadvantages of using fuzzy logic

- **Difficulties in creating the fuzzy rules base:**
  It is difficult to create the fuzzy rules base from input-output data if no fuzzy rule extraction technique is used
- **Accuracy of the inference depends directly to the number of fuzzy rules used in complex problem**
  The increase in input variables and fuzzy membership used will increase the number of fuzzy rules exponentially.

- **Number of fuzzy rules = M^I**
  where M = number of membership functions
  I = number of input variables
  (In case of complete rule base)

Ajánlott irodalom

- **The slides of this lecture are partially based on the books:**

Kóczy T. László és Tikk Domonkos: ***Fuzzy rendszerek*****,** Typotex Kiadó, 2000, **ISBN 963-9132-55-1**

J.-S. R. Jang, C.-T. Sun, E. Mizutani: ***Neuro-Fuzzy and Soft Computing*****,** Prentice Hall, 1997, **ISBN 0-13-261066-3**

Michael Negnevitsky: ***Artificial Intelligence: A Guide to Intelligent Systems*****,** Addison Wesley, Pearson Education Limited, 2002, **ISBN 0201-71159-1**