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An Extended Newsvendor Model for Customized Mass Production

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Abstract

Inventory control has been considered an essential problem in the management of supplier companies for several decades. In recent years numerous new supply chain and inventory control models have been developed to support management decisions. In this paper, we investigate the classical one-customer and one-supplier (news vendor) problem with an analytical, event-oriented model. Our basic aim is to extend the classical newsvendor model to *n* periods, which means that management decisions will be made only once at the beginning of a predefined time horizon based on demand forecast information. The developed model is also suitable to handle the product run-out problem which arises from the seasonality of the real demand. A new heuristic method was developed to determine the optimal length of the time horizon. We examine the problem by means of a simulation method and discuss the results.

Keywords: Supply Chain Management, Inventory Control, News Vendor Model, Stochastic Demand

1. Introduction

In the last 15 years, the business environment of companies in the field of mass production has altered. The demand rate for mass products has remained at a high level but numerous new requirements have appeared on the market. The life cycle of products is becoming ever shorter. Customer needs for stylish forms, new modern designs, special packaging or better product properties have greatly increased. Generally, mass production companies assemble and bundle their products from components originating from their supplier companies.

Changes in the business environment influence engineering and logistic relations between companies and suppliers. The former, simple buying-selling (so-called “cool”) relation has become much “warmer”. This means that cooperative and collaborative methods and activities have become the main object in SCM development. The fast evolution of IT technology plays an important role in this process. In many respects, real-time, network-similar collaboration of independent, locally-separated companies is not realizable without an effective computer network information system.

The whole productive-marketing chain of mass production is fairly long. The customer demands appear in shopping centres, which generate orders to logistical centres. Logistical centres transmit these demands to end-product manufacturers. End-product manufacturers forward orders to dozens of suppliers. This process generates on-floor orders (internal orders), starts production of lots, and places orders for raw material from suppliers. These multi-stage distributed information, decision and physical (producing and transporting) supply chains, material- and information-transmitter chains have a unavoidable delay, which directly leads to delays and instabilities, back orders, and overstock and becomes a source of unusable loss. Developing complex, large, collaborative supply systems necessitates increased information technology support of both business and technical processes. Complex ERP systems and auxiliary SCM modules and standalone SCM applications are available on the market to support the above-mentioned planning, decision, executive and information processes.

Relations of the marketing organizations, end-product manufacturers and supplier companies can be very complicated and various in practice. This motivates a wide examination of the available models and further investigation of effective decision supporting and planning methods.

If we analyse only the relation between the end-manufacturer and suppliers, even if strategy, tactical and operative collaborative areas can be separated, we find that strong fluctuation of stochastic market demands greatly influences the activity of mass production companies. Beyond the relationships of market competition, which is often stiff, care must be taken to maintain the obtained market positions. This generates a major emphasis on keeping order deadlines. The high requirements for readiness for delivery justify the realization of mixed, Make to Stock and Make to Order business policy.

In this paper we examine the possibility of supplier inventory policy in the case of non-deterministic demands. We assume that estimations concerning the future (forecasts) are solved; furthermore orders, acknowledgements, demands of delivery and the organization of the transport operation and synchronization of the planning process are also solved on the tactical level. We suppose that the supplier network on the strategy level is complete and bound by contract, and also that computerized communication conditions are available for the realization of business processes.

In this study (in the current stage of the research), we consider the problem as a relation of one supplier and one end-manufacturer. Moreover we assume that the supplier produces one product and he is in relation to one end-manufacturer. The end-manufacturer can give a demand forecast at least for the middle-term, namely several weeks in advance (or other predefined time periods). But concrete transport demand is given in the form of demand of delivery only a short time beforehand, typically one or two weeks. The information announced by the end-manufacturer is uncertain to some extent. The values of forecast and concrete demand of delivery naturally do not coincide at all times.

The primary objective of this research is to examine supplier inventory control policies, which ensure delivery completion (compliant Service Level: SL) in accordance with the demand of the end-manufacturer for a specific product, considering also the uncertainties. The production and inventory control policy of the supplier must be optimal in a sense, taking into consideration the collaborating relationship of the partners. In the first step, the above-mentioned policy must be characterized by time management (determination, control) of the inventory level. The end-product manufacturer has a contractual obligation to provide technical specifications, a long- and a middle-term forecast to the supplier. The supplier has a contractual obligation to provide services in time according to the agreement with the end-product manufacturer. The supplier can control the stock level through inner production and orders. The management task of the supplier is to determine what stock level should be maintained, and in what time and what size of series should management of the stock level be begun.

Concerning the restrictive conditions, we assume that the products produced and stored by the supplier are not perishable for the time of delivery on call and at the requisite time, and that the producing resources are available without constraint. The supplier needs raw materials to manufacture the products. We assume that these materials are available at the start time of production. We suppose that raw materials and finished goods are not in the same inventory, so the model does not deal with the potential question of the two stocks taking up much room from each other.

2. Related studies

Demand for efficiently modelling and solving inventory control problems, exists since establishment of industrial companies, factories and enterprises. First successful publications appeared at the beginning of the 1950's. Since then constantly numerous papers were published in the stockpiling area, which validates the up-to-datedness of the subject. Because the whole history overview will be diffuse, therefore only some name are emphasized now. The most important events related to the evolution of inventory control models are fully summarized in the paper of Hans-Joachim Girlich and Attila Chikán [1999]. The main line of direction of resource results is represented by one-product, one-period deterministic models. These models try to give an optimal policy in analytical way accordance with the objective function of the modeled reality. Multi-period deterministic and stochastic models applying more products were developed only in later years.

An another direction of the stockpiling policies is represented by game theory approaches. The reason of this should be found in the game theory itself, as in the "novelty" of the branch of mathematics. Game Theory assures effective methods to model the "warming-up" process of the supplier – end manufacturer, customer – vendor relation (becoming nowadays continually closer) and to modeling its cooperativity. In the following the results of the last near 50 years are surveyed.

In the late 1950's, the problem of "Optimal Inventory Policy" was analyzed by two important economists: Arrow [Arrow et al., 1951] and Marschak [Arrow et al, 1951]. Karlin's presentations solved this problem with her dynamic programming method („The Structure of Dynamic Programing Models") [Karlin, 1955]. Thirty-six years later, Alistair Milne [Milne, 1966] emphasized that one of the best papers in the area of production decisions and inventory analysis area was the study of Arrow, Karlin and Scarf entitled "Studies in the Mathematical Theory of Inventory and Production" [Karlin, 1958]. Among the deterministic models, the Wagner-Within method minimizing the total cost fills in a great role, which determines the optimal inventory level with O(n logn) calculation time for an *n* length finite time horizon. A remarkable scientist of the stockpiling theme is Herbert Scarf, who attained prominent results mainly at the game theory solutions. Since 1957 it was justified by his numerous publications, as well as his "*Computation of Equilibrium*" monograph, born in 1973.

E. Schneider's mathematical models deal with uncertainty-loaded problems of inventory control. E. Shaw in the "Elements of a Theory of Inventory" [Chikán, 1999] created a two-period uncertainty loaded model. In the 90's (S,s) type dynamic inventory control policies were published. The mathematician A. Markov laid strong foundations for the mathematical background for these models. In the meantime, John von Neumann and Oskar Morgenster's famous book, the "Theory of Games and Economic Behavior" [Neumann, 1940] became known, which gave a new direction to the approach of inventory problems. The paper of Dvoretzky, Kiefer and Wolfowitz [Dvoretzky et al., 1953] examined the (S,s) type policy in the case of a fixed time interval and penalty cost. Nowadays the analysis of inventory-holding problems has become an important part of the management of supply chains. Many excellent publications have been published related to this theme [Lal and Staelin (1984), Monohan (1984), Lee and Rosenblatt (1986), Dada and Srikanth (1987) and Weng (1995)], which work with the deterministic demand model [Girlich and Chikán, 1999].

In recent years further models have been published in the area of collaborative planning (Aviv), forecast and Vendor Management [Aviv and Federgruen, 1998], and information sharing within the supply chain [Gavirneni et al., 1999]. Nowadays in the explanation of the supply chain problems, the most prominent results are linked with the name of G.P. Cachon [Cachon, 1999, 2003]. For laying the foundation of the inventory policies, successful game theory results have sprung up.

In the large literature of inventory control models the so-called newsvendor model has a prominent role. The stochastic model is applied in a wide variety of areas of operations management for its simplicity and efficiency: centralized and decentralized supply chain inventory management (e.g., Shang and Song 2003, Cachon 2003), retail assortment planning (e.g., van Ryzin and Mahajan, 1999), international operations (e.g., Kouvelis and Gutierrez 1997), horizontal competition among firms facing stochastic demand (e.g., Lippman and McCardle, 1995), lead time competition (e.g., Li 1992), outsourcing and subcontracting decisions (e.g., Van Mieghem 1999), product and process redesign (Fisher and Raman 1996 and Lee 1996), and spot markets and inventory control (e.g., Lee and Whang 2002).

It is not allowed to pass over the classical EOQ (*Economic Order Quantity)* model, because it was widely applied since 1916 and its other modified variations are also used still at present. The model is strictly based on deterministic input-output condition scheme, but it is fairly indifferent to the estimation inaccuracy concerning the expected value of the demand.

Stockpiling in the management of supply chains plays an important role nowadays. With the rapid evolution of information technology, ERP (*Enterprise Resource Planning*) and SCM (*Supply Chain Management*) applications systems are gaining in significance. Dynamic systems with many products are manageable with operations research models or constraint programming methods. However, solutions based on analytical results and heuristics have a great part in “what if” type investigations and in the case of quick decisions.

3. Classical newsvendor problem in supply chains

In successful inventory control models, the critical inventory and the cost-optimal inventory policy are realized by decisions. In the course of these, decisions are made to determine starting time and quantity of the production. Naturally the individual decisions involve many responsibilities whose consequences appear in production, logistical and business costs. In the course of remaining non-reusable and overstocked products, stock finance and inventory holding costs occur. In addition, production of non-sufficient goods leads to penalty (back-order) costs. Modeling this latter case is a very difficult problem. Of course, different models can possess different objective functions, and by means of this a common interest can be realized between the end-product manufacturer and the supplier (common cost function). As a constraint the strict non-admittance of the “lack” (e.g. short cycle JIT) can be appeared. In this paper we consider a model that in general allows the risk of the back-order, but the frequency of these can be reduced to a small level by increasing the penalty costs.

The literature uses the concept of penalty costs in three different ways in inventory control policy. According to the first explanation, the supplier pays a penalty cost in the course of back-orders, which means undertaking the loss of the lost business of the end-manufacturer. This situation appears in a simple “cool” buying-selling relation.

In the second explanation the back-order occurs when the supplier increases the cost of the whole supply chain anyway, even if there is not any concrete lost business. Namely, the end-manufacturer should prevent the consequence of the supplier’s loss with more inner work, finished product reserves, reschedulings, etc. This approach infers some kind of “warm” (collaborative) relation among the business partners.

In the third explanation, not only the non-satisfied inner orders and the lost business but also the large (maybe never again saleable) stocks also cause a loss to the whole production chain, which must be suffered by all the partners jointly. This kind of close business, production and logistic relationship of the partners infers a long-term community of interest, some kind of “*Virtual enterprise*”. In this paper we refer to the explanation of penalty cost in accordance with the second interpretation.

This research supported by a consortium of six significant Hungarian mass production companies. The project was launched in 2005 and our team has been working on theoretical models of inventory controlling area. We examined the literature available on this field. We have chosen an event-oriented model for solving the inventory control problem of the supplier, which is based on cyclic demand of delivery and transport. This is the classical newsvendor model. The optimal stockpiling policy holds the supplier related costs at the minimal level on a long term. By minimizing costs, profit maximization can be indirectly attained. It is clear that in a cooperative supply system the supplier must tackle higher-level services. The end-manufacturer recompenses these with stable and reliable orders, with forecast of the processes, with detailed business and production information, and with long-term contractual safety. Of course, inventory policy of the supplier is only one (important) component of the cooperative (even collaborative) supply system. All elements of the whole relation system are not discussed here. We suppose the demand in the model is known as a time-dependent random variable with its expected value and distribution function (in the simplest cases it is uniform distribution in every *t* time between a pre-defined D_{min} and D_{max} value), which appears at the time of demand of delivery. The real demands arrive as pre-known, fixed periodicity to the supplier.

The element and the most important phases of the supplier inventory control policy are summarized as follows. Determination of the distribution function is needed for the first step of the method. In this step, we determine an optimal inventory level retailed on a long view, in terms of the costs. As a second step, we aim a lower inventory level than the optimal, using the forecast information. To achieve this, a critical inventory level must be determined, where the production and the non-production costs respecting to one time interval are equal. The third important step of the method is the decision itself, which is about starting a production cycle or not; i.e. if the current inventory level is less than the critical level, then production must be started, otherwise nothing is done until the next decision event.

Our model examines the problem from the point of view of the supplier. The collaborative interests appear in the model through the parameter values. After the solution of the model we review the steps performing the optimal policy.

3.1 Cost function for one-period

The classic newsvendor model considers a type of problem that many decision makers (newsvendors) encounter in the business world. Facing uncertain demands for limited-useful-life products (such as mobile phones, fashionable goods etc.), a decision maker (newsvendor) needs to decide how many units of these goods to order for a single selling period. Intuitively, if she/he orders too many (overage), this may cause unnecessary inventory cost. Thus, the cost will be too high. Whereas, if the decision maker (newsvendor) orders too few (underage), it will miss opportunities for additional profits because some customers have no chance to buy the goods. The optimal solution to this problem is characterized by a balance between the expected costs of overage and underage.

On the basis of the problem outlined in the model the supplier cost function regarding one time period can be formulated as a function of the parameters in the following manner [Hayriye, 2004]:

$$K(q) = c_f + c_v(q - x) + pE[max(D - q, 0)] + hE[max(q - D, 0)]. \qquad (1)$$

where the individual parameters are the following:

c_f – fixed cost. This cost always exists when the production of a series is started. [Ft / production]

c_v – variable cost. This cost type expresses the production cost of one product. [Ft / product]

p – penalty cost (or back order cost). If there is less raw material in the inventory than needed to satisfy the demands, this is the penalty cost of the unsatisfied orders. [Ft / product]

h – inventory and stock holding cost. [Ft / product]

D – This means the demand from the receiver for the product, which is an optional probability variable. [number / period]

E[D] – Expected value of the *D* stochastic variable.

F(D) – cumulative distribution function of *D*

f(D) – probability mass function of the demands *q* – The product quantity in the inventory. The decision of the inventory control policy concerns the product quantity in the inventory after the product decision. This parameter includes the initial inventory as well. If nothing is produced, then this quantity is equal to the initial quantity, i.e. concerning the existing inventory.

x – Initial inventory. We assume that the supplier possesses *x* products in the inventory at the beginning of the demand of the delivery period.

m – This means the effective producing quantity in the current time period. Its value is the difference between the optimal and the remaining quantity from the last time period.

The first part Equation (1) expresses that starting the production of each series carries some fixed costs, which express the starting cost of a new production cycle. The second part shows the variable costs of the products to be produced. Assuming that *x* product is available in the inventory, therefore one technological decision will result in the production of *m=q – x* products.

The third part of the cost function is the penalty cost (back-order), which symbolizes the costs issuing from unsatisfied demands. The *max(x,0)* function performing in the cost function will be different from zero if the demand is larger than the quantity in the inventory. Of course, in the real world there might be such practical cases when back-order is not allowable. This can be achievable via choosing a high *p* parameter. The last part of the equation expresses the holding cost, which arises at that time when the demand is less than the quantity of the end-product in the inventory. If the demand is more than the produced quantity of the end-product, then of course there are no additional charges, because the inventory will be empty after filling the orders.

On the basis of (1), determination of the optimal inventory level is a minimization problem [Hayriye, 2004][Mileff, 2006]. The solution method is not discussed here, [Hayriye, 2004] shows the details. So in the long run the amount of cost-optimal end-product can be calculated on the basis of the following relation:

$$F(S)=\frac{p-c_v}{p+h}=\frac{p-c_v}{(p-c_v)+(h+c_v)}\text{, thus} \qquad (2)$$

$$F(S)=\frac{shortage\ \ cost}{(shortage\ \ cost)+(surplus\ \ cost)}\longrightarrow P(D\le q)=\frac{P(D\le q)}{P(D\le q)+P(D>q)}, \qquad (3)$$

where *F()* is the distribution function of the demand. The value of S expresses the amount of the end-product that should be in the inventory when the demand appears. In the optimality condition $\xi := (p-c_v)/(p+h)$ is known as the critical fraction. Also note that the critical fraction is the probability of not stocking out (somewhere known as *Cycle Service Level*(CSL) [Porteus, 2001]). So $\xi = P(Demand \le q)$. According to the definition it is the probability that the supplier can satisfy every demand during the time horizon. Characteristic of CSL can be seen at figure 1.

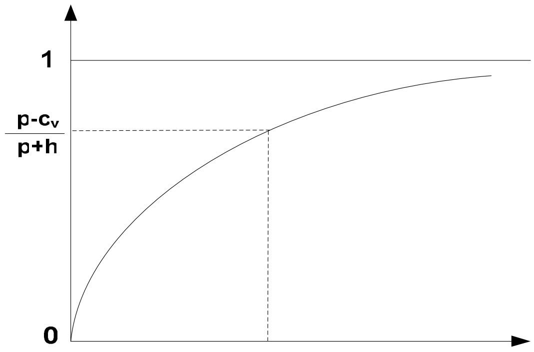

Figure 1. CSL characteristic

It can be noticed that if $p < c_v$, then the problem cannot be explained by mathematics. Since at this time $F(S) < 0$, which the definition of the distribution function does not allow. This means in reality that penalty cost occurring at the non-production is less than occurring at the production. From this reason supplier will produce nothing, rather suffers the consequence of the back-order.

It is easy to see that if there are no available products in stock to satisfy the demand, then necessarily a production cycle should not be started, because it carries fixed costs which make the production of a small volume expensive. Consequently it is conceivable that there definitely exists a critical amount, which is smaller than the optimal (*S*) amount, but by choosing this quantity it is more profitable to sustain the risk of the back-order. The name of that point where the cost of the decision about producing or non-producing (as we would rather undertake the risk of the back-order) is equal is the critical inventory level (*s*). The critical level is probably smaller than the long-term cost-optimal inventory level. If the products in the inventory are less than this, only then is it profitable to increase the stock in hand to the optimal level [Mileff and Nehéz, 2006]. However, using the critical inventory is a collaborative question which depends on the contractual relationship among the partners. In literature this approach of inventory control is known as (*S,s*) policy and at first time was mentioned by Herbert Scarf. In this paper this method will not be demonstrated, but it is shown properly in [Hayriye, 2004] and [Mileff, Nehéz, 2006].

4. Extending the news vendor model to more joint weeks

Numerous examples exists in the literature solving the multi-period inventory problems. The study [David Simchi-Levi, Julie L. Swann et al, 2004] discusses many of these methods and solutions. These methods generally solve multi-period problems with the help of dynamic programming, stochastic programming or with some kind of searching algorithms. The solution effort of these, are extremely time consuming and it can be computed with difficulties. The heuristic approaches always apply model simplification to solve the NP hard multi-period problems. Therefore our aim was to develop a model, based on the original newsvendor model to solve multi-period inventory decision problems in case of customized mass production.

The optimal inventory policy of the supplier was discussed above in detail projected for one production time horizon. Here, this policy will be complemented in such a way that it is to be suitable to cover uniformly *n* number production cycles (weeks). Then we demonstrate the determination of the optimal inventory level in the course of the joint production of *n* number of weeks. With the help of these optimum values it becomes possible to determine the necessary production cycles in a specific time horizon. This optimal cycle number, which means the minimal setup number, opens the door to minimize the costs on the supplier's side.

4.1 Determining the inventory optimum for n joint weeks

Let us extend the News Vendor problem to *n* joint weeks. This means that we do not produce every week but we try to determine an optimal number of *n* joint weeks based on demand forecast information. Especially in mass production, where setup costs and production rate are high values and holding cost is low, it is not worth producing every week. We suppose that necessary production capacities are available.

Nevertheless holding and penalty costs can arise each week, thus with these factors the new cost function becomes quite complicated. The cost function of the joint production for *n* weeks is the following:

$$K_{123\ldots n}(q_{123\ldots n}) = c_f + c_v\left[q_{123\ldots n} - x\right] + hE\left[q_{123\ldots n} - D_1\right]^+ + hE\left[q_{123\ldots n} - D_1 - D_2\right]^+ + \ldots +$$
$$+ hE\left[q_{123\ldots n} - D_1 - D_2 - \ldots - D_n\right]^+ + pE\left[D_1 - q_{123\ldots n}\right]^+ + pE\left[(D_1 + D_2) - q_{123\ldots n}\right]^+ + \ldots + \qquad (4)$$
$$+ pE\left[(D_1 + D_2 + \ldots + D_{n-1}) - q_{123\ldots n}\right]^+ + pE\left[D_n + \left[\ldots + \left[D_2 + \left[D_1 - q_{123\ldots n}\right]^-\right]^-\right]^-\right]^+,$$

where $q_{123\ldots n}$ means the quantity which appears in the supplier inventory early in the cycle for *n* week after production. $D_1, D_2, \ldots, D_n$ mean the demand of the specific week, which appears as a random variable. It can be seen that the last part of the function is fairly complicated, but by means of several mathematical transformations (which are not discussed in this paper) we obtain a simpler variation of (4):

$$K_{123\ldots n}(q_{123\ldots n}) = c_f + c_v\left[q_{123\ldots n} - x\right] + hE\left[q_{123\ldots n} - D_1\right]^+ + hE\left[q_{123\ldots n} - D_1 - D_2\right]^+ + \ldots +$$
$$+ hE\left[q_{123\ldots n} - D_1 - D_2 - \ldots - D_n\right]^+ + pE\left[(D_1 + D_2 + \ldots + D_n) - q_{123\ldots n}\right]^+. \qquad (5)$$

The difference between (4) and (5) is that the penalty cost was simplified because we exploited that fact the penalty is considered as an independent value during *n* periods. Therefore penalty cost can be calculated as a difference in the sum of demands and production quantity.

From the form of (5) it can be expressly seen that the problem is traced back to the one-week production problem. The solution, the optimal inventory quantity of the joint production for *n* weeks, arises as a minimization problem in the following manner:

$$F_{123\ldots n}(q_{123\ldots n}*) = \frac{p - c_v - hF_1(q_{123\ldots n}) - hF_{12}(q_{123\ldots n}) - hF_{123}(q_{123\ldots n}) - \ldots - hF_{123\ldots n-1}(q_{123\ldots n})}{h + p}, \qquad (6)$$

where $F()$ represents the joint distribution function in compliance with the number of weeks drawn together. The $q_{123\ldots n}^*$ - which satisfies the equation - expresses that the finished goods must be in the inventory at the time when customer demand appears with regard to *n* weeks. (6) can be solved by using numerical methods but in special cases some simplification can be applied.
In practical calculations, values of $hF_1(q_{123\ldots n}), hF_{12}(q_{123\ldots n}), hF_{123}(q_{123\ldots n}), \ldots, hF_{123\ldots n-2}(q_{123\ldots n})$ can be approximated by 1. Note that: assuming that *F1(x) is a* cumulative distribution function of a uniform distribution. If argument *x* is greater than the maximum value of the given uniform distribution, then *F(x)* always gives 1 by definition. This way equation (6) becomes a simplified form as follows:

$$F_{123\ldots n}(q_{123\ldots n}*) = \frac{p - c_v - (n-2)\cdot h - hF_{123\ldots n-1}(q_{123\ldots n})}{h + p}. \qquad (7)$$

The numerator can be a negative value in that case, when holding cost during the weeks is greater than a certain limit and in this case, naturally there is no optimal solution. This time the number of weeks have to be reduced, because it is cheaper if the supplier does not produce anything.
Of course the critical inventory mentioned for one-week production can be applied in the case of joint production for *n* numbers of production cycles too. However this is not discussed in this paper.

4.2 Optimal inventory control policy for n weeks

As we mentioned above, the optimal inventory quantity in case of joint production for *n* weeks can be calculated with the relation (6) showing similar symmetry to the one-week production problem. In the following, the practical importance and of joint production for *n* weeks will be introduce in brief through a concrete example. The empirical factors show that the (high) value of the production fixed costs influences significantly the number of setups in the specific production time horizon and indirectly the costs. The greater this value is, the less profitable production cycle usage is in the specific order time window. Thus cost referring to the specific time horizon will be minimal if and only if the number of setups is minimal as a function of production fixed costs. In the following, both the optimal number of setups and the optimal production quantity can be determined. The method uses directly the above-mentioned cost function of joint production for *n* weeks.

We assume that the distribution functions of the probability random variables in the cost function are uniform. The manner can be applied in case of optional distribution, but in this example we choose normal distribution because of easier lucidity. The reason is that the distribution and cumulative functions of the new variable (e.g.: $D_{123} = D_1+D_2+D_3$) established from the sum of uniform distribution probability random variable are complicated. Supposing normal distribution, the σ and μ values of this new variable is realized as the sum of the σ and μ values of the added variables. Moreover we assume that values of σ and μ are equal for all weeks.

The basic idea of the method is as follows: with the help of the cost function (6), the optimal production quantity can be determined regarding an optional time horizon. However this quantity is independent from production fixed costs, because the *n* number of weeks are considered as a production cycle. It is easy to see that relating to a certain time horizon the necessary number of setups depends greatly on the value of production fixed costs. Controlling this problem, we introduce the concept of per-unit cost, which means the cost per unit regarding a specific time horizon.

Let denote the value of per-unit cost $\hat{K}_i = \frac{K_i}{q_i^*}$, where K_i means the cost of *i* th number of jointly produced weeks, q_i^* is the optimal quantity of *i* number of jointly produced weeks and $i = 1...k$. We suppose that the value of per-unit cost in case of different numbers of jointly produced weeks will be different. The objective is to find the minimal from this set. Thus finding the $\hat{K}_{min}$, which satisfies the following equation: $\hat{K}_{min} = min\{\hat{K}_1, \hat{K}_2, ..., \hat{K}_n\}$. Possessing the minimum shows clearly the necessary number of weeks in a specific time horizon needed for joint production. The necessary minimal number of setups arises already from this.

The first step of the method is using the formula of joint production for *n* weeks. In the course of this the optimum of *k* number of jointly produced weeks comes to determination. Experience shows that the value of *k* can be maximum 7 – 8 in practice. The determination of the optimum can be calculated analytically with the help of the above-mentioned method, which needs very little calculation time. We prove our assumption through the next practical example.

Distribution of demands for products are normal with 15/week mean value and with variation $\sigma = 3$. The back-order cost is *p* = 40 unit for each element. Holding cost is *h* = 2 unit/period. The fixed cost of the production is c_f = 120 unit/series. Let the value of variable cost c_v = 5 unit. The per-unit costs of the $k = 1...9$ number of jointly produced weeks and the produced optimal *q* quantities obtained in the course of computations are summarized in Table 1.

Table 1. Determining the optimal week number for joint production

	Week 1	***Week 2***	***Week 3***	***Week 4***	***Week 5***	***Week 6***	***Week 7***
Optimal quantity	16.91364	*33.26343*	***49.16258***	*64.67296*	*79.83145*	*94.66017*	*109.17202*

Per-unit Cost	*12.68306*	*9.93307*	***9.78449***	*10.21825*	*10.86718*	*11.61334*	*12.40708*

The columns of the table mean the number of jointly produced weeks. The first row denotes the optimal quantity as a function of jointly produced weeks and the last row means the per-unit cost. For example, 79.83145 in the first row means that if we produced on the basis of 5-weeks forecast information, the optimal quantity would be 79.83145. In this case, 10.86718 is the per-unit cost of the production. Clearly, the per-unit cost is minimal in the third column (9.78449). With the help of Figure 1, per-unit cost can be studied as function of jointly produced weeks.

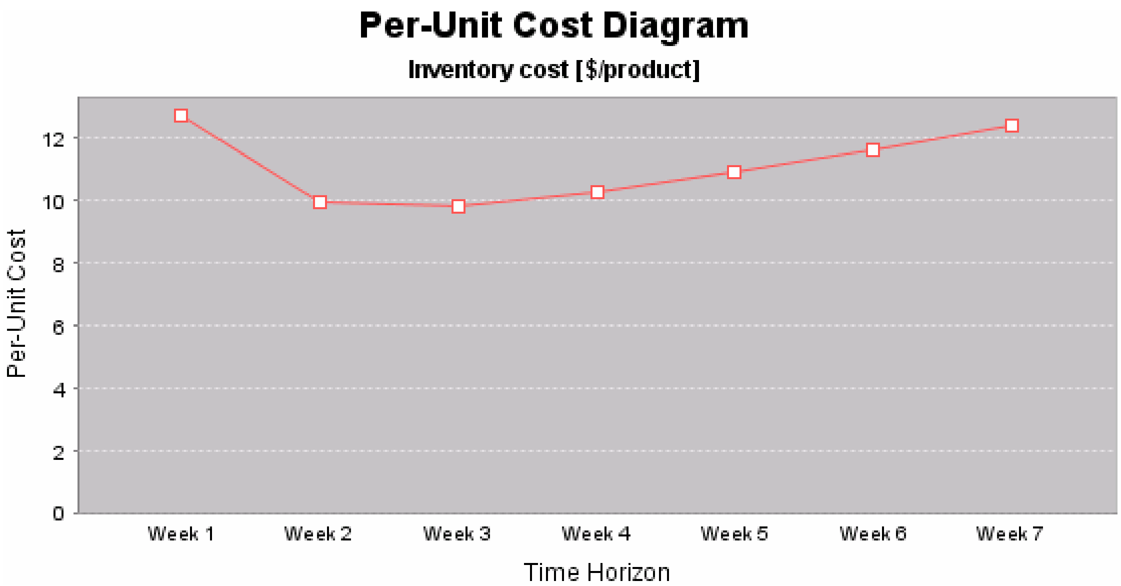

Figure 2. Variation of per-unit cost

The *x* axis in Figure 1. shows the production time horizon as function of weeks and the *y* axis represents the per-unit cost as function of jointly produced weeks. Once again, it is easy to see that the value of per-unit cost is minimal supposing joint production of a three-week (cycle). Using this minimal cycle number the costs will be really minimal.

Using this method the minimal number of required production cycles can be determined in a specific production time horizon, as can be the optimal quantity that needs to be manufactured in the cycle. The accuracy of our heuristic was verified using the method of constraint programming and with a genetic algorithm. Results in Table 2 prove clearly the efficiency of the new method. The enormous advantage of this manner is that it is substantially faster than brute force or a genetic algorithm. The measurement of execution times can be seen in Table 1. Since it is an analytical solution, therefore this will be always an exact solution or at least the error can be estimated in advance.

Table 2 shows the execution times of three different methods which are summarized applying the above mentioned parameters. The objective is to determine the optimal policy of a seven week long time horizon. This means that: methods establish the number of weeks, when production is necessary. Of course, in case of production the optimal quantity is computed as well.

Table 2. Comparison of execution times

Methods	**execution time (m. sec)**						
	Week 1	**Week 2**	**Week 3**	**Week 4**	**Week 5**	**Week 6**	**Week 7**
CP	200	600	9000	627000	n.a.	n.a.	n.a.
Genetic	200	400	700	1000	2000	3000	5000

Heuristic	10	10	20	50	50	50	50

Note that. n.a. means that execution time was greater than 30 min, therefore we skipped the test. Method CP symbolizes the basic constraint programming method. First row of Table 2 shows that this method cannot be used for solving n-week problems, because it is an NP hard problem in this approach. The values of the second row belong to the genetic algorithm solution. Genetic algorithm approach is faster than basic constraint programming solution, but the obtained results were only quasi-optimal solutions in some instances, which approximate the value of the true optimum in 99%. Values in the last row indicate well the efficiency of the newsvendor extension and of the heuristic method. Because the method makes analytical computing possible, so execution time will be extremely short.

5. Interpretation of product run-out

Following the explanation of joint production of production cycles we change over to the run-out problem of the products, which has nowadays a great importance. Market estimations prove clearly that natural fluctuation can be observed at product demands. It means that a kind of so-called seasonality can be explained for products. According to this seasonality, demands for products change greatly in a certain period. Almost all larger or smaller commercial firms, retailers, suppliers come face to face with this problem. They try to defend themselves against this with help of human intuition and experience of previous years.

Of course human intelligence is essential in case of a complex problem. Our aim is to extend the model with new elements, which can help to solve the problem beside human intuition. Applying this extension the more exact computation of necessary inventory level becomes possible.

The problem of product run-out is examined in case of pre-mentioned joint production of production cycles. The terminology “run-out” expresses that the demand for a specific product will be reduced to zero after a certain time. There was given any notice of the demand from the forecast system, but the product suddenly ran out (E.g. packaging material produced for Christmas). It is looked for an element (function), which expresses this in time ascending risk factor. In compliance with this, the run-out of demands is modelled with a random variable with Poisson distribution function. In the following figure the Poisson distribution function can be seen in case of $\lambda = 15$.

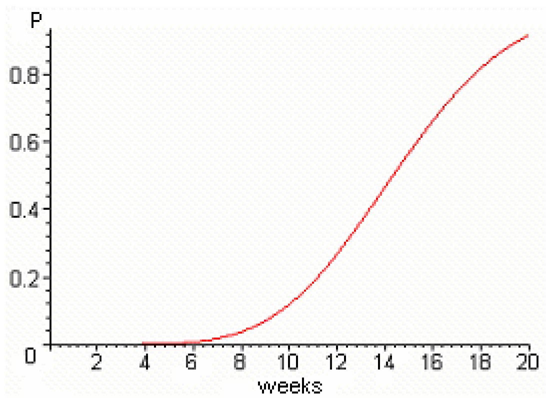

Figure 3. Poisson distribution function with $\lambda = 15$.

$\lambda = 15$ means that the event occurs once during 15 time period (week). Figure 3 shows that the probability of occurrence during the first 8 weeks is small ($P(\zeta < 8) \approx 0.066$). So the product runs out during the first 8 weeks with 6.6% probability.

Applying two weeks joint production, the cost function with its introduced new parts can be represented as follows:

$$K_{12}(q_{12}) = c_f + c_v[q_{12} - x] + hE[q_{12} - D_1]^+ + hE[q_{12} - D_{12}]^+ + pE[D_{12} - q_{12}]^+ + \\ + dR(1,\lambda)E(q_{12} - D_1)^+ + dR(2,\lambda)E(q_{12} - D_{12})^+ \quad , \tag{8}$$

Where *d* is a positive real number and expresses the loss per unit of left over goods. $R(i,\lambda)$ is the Poisson distribution function, where *i* is a positive integer designating the weeks. $\lambda = [0..\infty]$ is the parameter of the Poisson distribution.

On the basis of the above mentioned expressions:

$$\frac{dK(q_{12})}{dq_{12}} = c_v + hF_1(q_{12}) + hF_{12}(q_{12}) - p(1 - F_{12}(q_{12})) + dR(1,\lambda)F_1(q_{12}) + dR(2,\lambda)F_{12}(q_{12}).$$

The derivation is similar to the other solution in the appendix, so this is not discussed here. Thus the solution:

$$F_{12}(q_{12}*) = \frac{p - c_v - hF_1(q_{12}) - dR(1,\lambda)F_1(q_{12})}{h + p + dR(2,\lambda)}. \tag{9}$$

The problem can be explained similarly in case of N week joint production as well. Product run-out is interpreted for *n* number of weeks. The solution in this case: (It is not detailed here, but can be proved easily)

$$F_{123...n}(q_{123...n}*) = \frac{p - c_v - hF_1(q_{123...n}) - hF_{12}(q_{123...n}) - ... - hF_{123...n-1}(q_{123...n}) - \sum_{i=1}^{n-1}[dR(i,\lambda)]}{h + p + dR(n,\lambda)}. \tag{10}$$

In favour of practical feasibility, choosing values of *d* and λ have crucial importance. Value of *d* can be comprehended as penalty cost, which changes per units. λ expresses the seasonality of the product. So the smaller is its value, the larger is the danger of its run-out.

Conclusions

In the present paper we examined the problem in case of the collaborative relation of one supplier and one customer on the basis of claim of a Hungarian mass-production company. Improving the well-known models from the literature, we optimized the cost function of the supplier as a function of the parameters, which does not eliminate the possibility of back-order and is applicable for an optional time horizon. We extended the problem of a one-week production cycle to become the production of optional, *n* number of jointly produced weeks possible. Understanding the problem as a non-linear optimization problem, we determined the optimal inventory level. With the help of a heuristic method over and above the optimal inventory level, the minimal required number of jointly produced weeks can be defined in an exact way. The efficiency of the method was proved by a genetic algorithm and constraint programming. The simulation results show clearly that calculation time of the method is small; therefore it is suitable for fast testing of different policies and decision alternatives.
The developed model gives also a suitable solution to handle the product run-out problems as well, which arises from the seasonality of the real demand.
Henceforth we aim to extend the developed model to be applicable multi-product problems with natural capacity constraints. Part of our aim is to integrate the forecast information of the expected demand into the model, as well as to examine how historical data and uncertain forecasts influence the conformation of inventory level over time.

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The following complicated derivations in the appendices have been verified by a symbolic mathematical software package.

APPENDIX 1

Determining the optimal inventory level of a one-week production cycle

The cost function of the problem outlined in the model can be defined as a function of the parameters in the following manner:

$$K(q) = c_f + c_v(q-x) + p[max(D-q,0)] + h[max(q-D,0)].$$

The question is how much the cost for one period will be, so how much the expected value of the costs will be: $g(q)=E[K(q)]$. From this, formula (1) is obtained. Let denote the cumulative distribution function of the demands with *F(x)* and let *f(x)* be the probability mass function of the demands, so $F(x) = P(D \le x) = \int_0^x f(y)dy$. We assume that *f(x)* is continuous in $[0,\infty)$ in the following proof. The conclusion in this section still holds when *D* is a general continuous random variable.

The following relations are true in case of any *q* >0:

$$E(q-D)^+ = \int_0^\infty (q-x)^+ f(x)dx = \int_0^\infty max(q-x,0)f(x)dx = \int_0^q (q-x)f(x)dx = q\int_0^q f(x)dx - \int_0^q xf(x)\,dx,$$

$$E(D-q)^+ = \int_0^\infty (x-q)^+ f(x)dx = \int_0^\infty max(x-q,0)f(x)dx = \int_q^\infty (x-q)f(x)dx = -q\int_q^\infty f(x)dx + \int_q^\infty xf(x)\,dx,$$

where $(q-D)^+ = max(q-D,0)$, and $(D-q)^+ = max(D-q,0)$.

With these two relations the cost function will be the following:

$$g(q) = c_f + c_v(q-x) + h\left[q\int_0^q f(x)dx - \int_0^q xf(x)dx\right] + p\left[-q\int_q^\infty f(x)dx + \int_q^\infty xf(x)dx\right].$$

Our objective is to find the minimal *g*, so we must find a *q* which satisfies the $g'(q)=0$. For derivation we use these fundamental rules:

$$\frac{d}{dx}\int_0^x h(t)dt = h(x), \text{ and } \frac{d}{dx}\int_0^\infty h(t)dt = -h(x).$$

Thus:

$$g'(q) = c_v + h\left[\int_0^q f(x)dx + qf(q) - qf(q)\right] + p\left[-\int_q^\infty f(x)dx - qf(q) + qf(q)\right],$$

$$g'(q) = c_v + h\left[\int_0^q f(x)dx\right] - p\left[\int_q^\infty f(x)dx\right].$$

After derivation using the fundamental rules we obtain the following relation:

$$g'(q) = c_v + hF(q) - p(1 - F(q)).$$

Setting $g'(q) = 0$, we find that q^* must satisfy

$$F(q^*) = \frac{p - c_v}{p + h}.$$

To check whether g has a unique maximum, we take the second derivative of g,

$$g''(q) = (p + h)f(q).$$

Examining the result it is easy to see that $p + h \geq 0$ without exception, because both p and h are positive integers. Thus $g''(q) \geq 0$, so g is a convex function on $[0, \infty)$.

APPENDIX 2

Determining the optimal inventory level of a two-week joint production cycle

The following equation expresses the two week cost function:

$$K_{12}(q_{12}) = c_f + c_v[(q_1 + q_2) - x] + hE[(q_1 + q_2) - D_1]^+ + hE[(q_1 + q_2) - D_1 - D_2]^+ + pE[D_1 - (q_1 + q_2)]^+$$
$$+ pE\left[D_2 + [D_1 - (q_1 + q_2)]^-\right]^+.$$

It is easy to see that the function has been completed with the parts regarding to the second week. To simplify the function, some formulas are applied as follows:

$$min\{D, q\} = D - (D - q)^+,$$

$$max\{D, q\} = D^+,$$

$$min\{D, 0\} = D^- = D - D^+.$$

The last part of the equation is fairly complicated, so using these formulas it can be transformed to a more simple form:

$$D_2 + [D_1 - (q_1 + q_2)]^- = D_2 + \left[D_1 - (q_1 + q_2) - [D_1 - (q_1 + q_2)]^+\right].$$

Assume that the demand of the second week is independent of the demand occurring at the first week. So demands can be explained as independent probability random variables, therefore using the above formulas the expected value of the last part can be separated to two different parts. Penalty cost can be computed separately for every week and the joint expected penalty cost of the two weeks equals to the sum of the expected penalty costs of the certain weeks. So in the following we use the joint cumulative distribution and probability mass function of the merged probability random variables.

On the basis of the formulas and these remarks, the cost function changes as follows:

$$K_{12}(q_{12}) = c_f + c_v[q_{12} - x] + hE[q_{12} - D_1]^+ + hE[q_{12} - D_{12}]^+ + pE[D_1 - q_{12}]^+ + pE[D_{12} - q_{12}]^+ -$$
$$- pE[D_1 - q_{12}]^+ .$$

In a more simple form:

$$K_{12}(q_{12}) = c_f + c_v[q_{12} - x] + hE[q_{12} - D_1]^+ + hE[q_{12} - D_{12}]^+ + pE[D_{12} - q_{12}]^+ ,$$

where $q_{12} = q_1 + q_2$ and $D_{12} = D_1 + D_2$.

Note: q_{12} means the jointly produced quantity for two weeks, which is realized by the sum of producing quantities. In favour of more simplification D_{12} indicates the sum of the demands projected to the certain weeks. Because demands can be explained as independent probability random variables, therefore they can be summarized, and used jointly.

Applying the definition of the expected value:

$$K_{12}(q_{12}) = c_f + c_v[q_{12} - x] + h\left[q_{12}\int_0^{q12} f_1(x_1)dx_1 - \int_0^{q12} f_1(x_1)dx_1\right] + h\left[q_{12}\int_0^{q12} f_{12}(x_{12})dx_{12} - \int_0^{q12} x_{12} f_{12}(x_{12})dx_{12}\right]$$
$$+ p\left[\int_{q12}^{\infty} x_{12} f_{12}(x_{12})dx_{12} - q_{12}\int_{q12}^{\infty} f_{12}(x_{12})dx_{12}\right].$$

The objective is to keep the costs at the minimum level finding the minimum of the cost function. Performing the derivation in accordance with the jointly producing quantity, we get the equation:

Simplified the equation:

$$\frac{dK(q_{12})}{dq_{12}} = c_v + h\int_0^{q12} f_1(x_1)dx_1 + h\int_0^{q12} f_{12}(x_{12})dx_{12} - p\int_{q12}^{\infty} f_{12}(x_{12})dx_{12} .$$

Using the above mentioned rules:

$$\frac{dK(q_{12})}{dq_{12}} = c_v + hF_1(q_{12}) + hF_{12}(q_{12}) - p(1 - F_{12}(q_{12})).$$

The theory of extreme value calculation declares, if $K'(q_{12}) = 0$ then there is a $q_{12}*$, which satisfies the following equation:

$$F_{12}(q_{12}*) = \frac{p - c_v - hF_1(q_{12})}{h + p} .$$

The second derivate of the function in accordance with q_{12}:

$$\frac{dK(q_{12})}{d^2 q_{12}} = hf_1(q_{12}) + hf_{12}(q_{12}) + pf_{12}(q_{12}) .$$

Verifying the result it can be seen that $hf_1(q_{12}) + hf_{12}(q_{12}) + pf_{12}(q_{12}) \geq 0$ in every case, because both *p* and *h* are positive integers, and the values of each probability mass function belonging to the certain weeks are positive integers. Thus $K''(q_{12}) \geq 0$, namely K_{12} is convex on $[0, \infty)$ interval.

APPENDIX 3

Determining the optimal inventory level of an n-week joint production cycle

The initial point to determine the optimal inventory level for n joint week is the n-week cost function similarly to in what has gone before:

$$K_{123...n}(q_{123...n}) = c_f + c_v\left[q_{123...n} - x\right] + hE\left[q_{123...n} - D_1\right]^+ + hE\left[q_{123...n} - D_1 - D_2\right]^+ + ... +$$
$$+ hE\left[q_{123...n} - D_1 - D_2 - ... - D_n\right]^+ + pE\left[D_1 - q_{123...n}\right]^+ + pE\left[(D_1 + D_2) - q_{123...n}\right]^+ + ... +$$
$$+ pE\left[(D_1 + D_2 + ... + D_{n-1}) - q_{123...n}\right]^+ + pE\left[D_n + \left[... + \left[D_2 + \left[D_1 - q_{123...n}\right]^-\right]^-\right]^-\right]^+$$

Where $q_{123...n} = q_1 + q_2 + ... + q_n$ and $D_{123...n} = D_1 + D_2 + ... + D_n$. The last part of the equation, which expresses the back-order cost, requires a transformation because of its complexity. We use the formulas mentioned at the two-week policy. As a result of these the transformed cost function:

$$K_{123...n}(q_{123...n}) = c_f + c_v\left[q_{123...n} - x\right] + hE\left[q_{123...n} - D_1\right]^+ + hE\left[q_{123...n} - D_1 - D_2\right]^+ + ... +$$
$$+ hE\left[q_{123...n} - D_1 - D_2 - ... - D_n\right]^+ + pE\left[(D_1 + D_2 + ... + D_n) - q_{123...n}\right]^+$$

The difference between these two equations comes from handling the penalty cost. Simplification arises from the assumption that demands of certain weeks, represented as probability random variable with optional distribution function, are independent during the whole time horizon. In this comprehension penalty cost can be computed as a difference of the summarized demands for n weeks and the current inventory level. Applying the definition of the expected value:

$$K_{123...n}(q_{123...n}) = c_f + c_v\left[q_{123...n} - x\right] + h\left[\begin{array}{l}\int_0^{q_{123...n}} q_{123...n} f_1(x_1)dx_1 - \\ - \int_0^{q_{123...n}} x_1 f_1(x_1)dx_1\end{array}\right] + h\left[\begin{array}{l}\int_0^{q_{123...n}} q_{123...n} f_{12}(x_{12})dx_{12} - \\ - \int_0^{q_{123...n}} x_{12} f_{12}(x_{12})dx_{12}\end{array}\right] +$$
$$+ ... + h\left[\begin{array}{l}\int_0^{q_{123...n}} q_{123...n} f_{123...n}(x_{123...n})dx_{123...n} - \\ - \int_0^{q_{123...n}} x_{123...n} f_{123...n}(x_{123...n})dx_{123...n}\end{array}\right] + p\left[\begin{array}{l}\int_{q_{123...n}}^{\infty} x_{123...n} f_{123...n}(x_{123...n})dx_{123...n} - \\ - \int_{q_{123...n}}^{\infty} q_{123...n} f_{123...n}(x_{123...n})dx_{123...n}\end{array}\right]$$

The objective is again to keep the costs at the minimum level and to find the minimum of the cost function. Performing the derivation in accordance with the jointly producing quantity $q_{123...n}$, we get the equation:

$$\frac{dK_{123..n}(q_{123..n})}{dq_{123..n}} = c_v + h\left[\int_0^{q_{123..n}} f_1(x_1)dx_1 + q_{123..n}f_1(q_{123..n}) - q_{123..n}f_1(q_{123..n})\right] + h\left[\begin{array}{l}\int_0^{q_{123..n}} f_{12}(x_{12})dx_{12} + q_{123..n}f_{12}(q_{123..n}) - \\ -q_{123..n}f_{12}(q_{123..n})\end{array}\right] +$$

$$+ \ldots + h\left[\begin{array}{l}\int_0^{q_{123..n}} f_{123..n}(x_{123..n})dx_{123..n} + q_{123..n}f_{123..n}(q_{123..n}) - \\ -q_{123..n}f_{123..n}(q_{123..n})\end{array}\right] + p\left[\begin{array}{l}q_{123..n}f_{123..n}(q_{123..n}) - \int_{q_{123.n}}^{\infty} f_{123..n}(x_{123..n})dx_{123..n} \\ -q_{123..n}f_{123..n}(q_{123..n})\end{array}\right]$$

Simplified:

$$\frac{dK_{123...n}(q_{123...n})}{dq_{123...n}} = c_v + h\left[\int_0^{q_{123...n}} f_1(x_1)dx_1\right] + h\left[\int_0^{q_{123...n}} f_{12}(x_{12})dx_{12}\right] + \ldots + h\left[\int_0^{q_{123...n}} f_{123...n}(x_{123...n})dx_{123...n}\right] -$$

$$- p\left[\int_{q_{123...n}}^{\infty} f_{123...n}(x_{123...n})dx_{123...n}\right]$$

Applying the above mentioned rules the equation gets the form:

$$\frac{dK_{123...n}(q_{123...n})}{dq_{123...n}} = c_v + hF_1(q_{123...n}) + hF_{12}(q_{123...n}) + \ldots + hF_{123...n}(q_{123...n}) - p\left(1 - F_{123...n}(q_{123...n})\right).$$

The theory of extreme value calculation declares, if $K'(q_{123...n}) = 0$ then there is a $q_{123...n}*$, which satisfies the following equation:

$$F_{123...n}(q_{123...n}*) = \frac{p - c_v - hF_1(q_{123...n}) - hF_{12}(q_{123...n}) - \ldots - hF_{123...n-1}(q_{123...n})}{h + p}.$$

To verify that $q_{123...n}*$ is a real minimum of the function the second derivate helps. Thus

$$\frac{dK_{123...n}(q_{123...n})}{d^2q_{123...n}} = hf_1(q_{123...n}) + hf_{12}(q_{123...n}) + \ldots + hf_{123...n}(q_{123...n}) + pf_{123...n}(q_{123...n})$$

Verifying the result it is easy to see that $hf_1(q_{123...n}) + hf_{12}(q_{123...n}) + \ldots + hf_{123...n}(q_{123...n}) + pf_{123...n}(q_{123...n}) \geq 0$ in every case, because both *p* and *h* are positive integers, and the values of each probability mass function belonging to the certain weeks are positive integers. Thus $K''(q_{123...n}) \geq 0$, namely $K_{123...n}$ is convex on $[0, \infty)$ interval.