Intelligens Számítási Módszerek
Fuzzy halmazok, műveletek Fuzzy halmazokon

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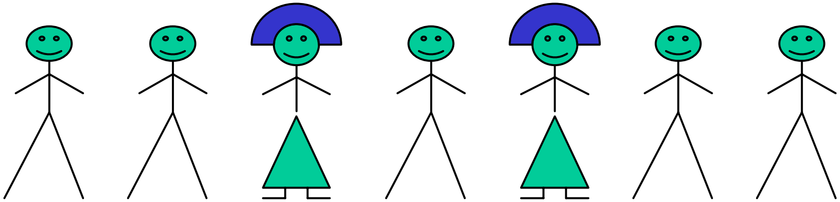
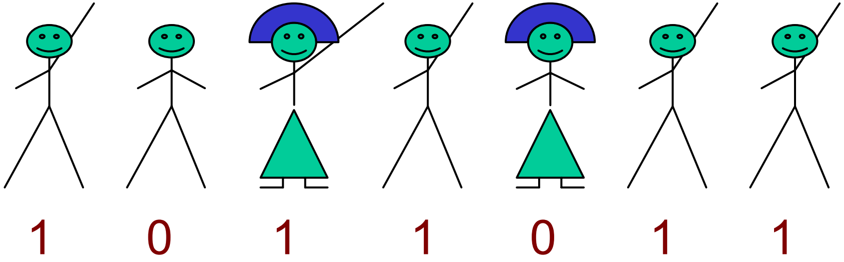
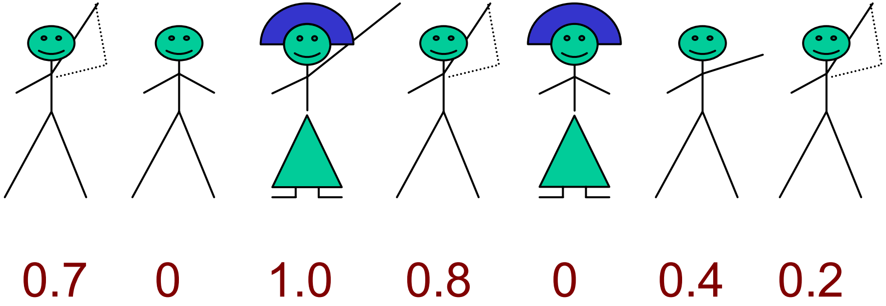
Fuzzy set



- **A class of students**
  (E.g. M.Sc. Students taking „Fuzzy Theory”)
  The universe of discourse: X



- **“Who does have a driver’s licence?”**
  A subset of X = A (Crisp) Set
  χ(X) = CHARACTERISTIC FUNCTIO



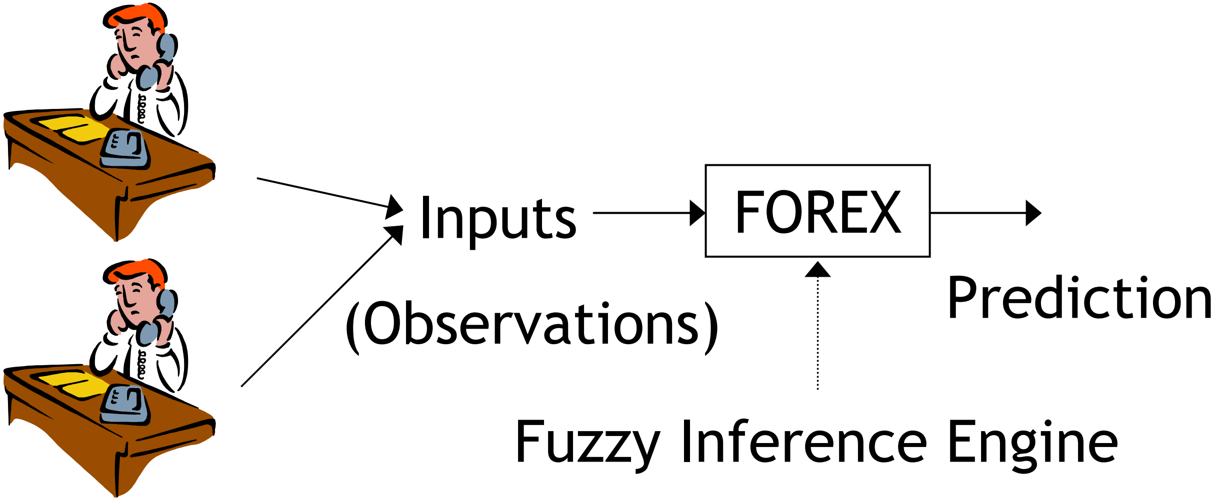
- **“Who can drive very well?”**
  μ(X) = MEMBERSHIP FUNCTION

History of fuzzy theory

- **Fuzzy sets & logic: Zadeh 1964/1965-**
- **Fuzzy algorithm: Zadeh 1968-(1973)-**
- **Fuzzy control by linguistic rules: Mamdani & Al. ~1975-**
- **Industrial applications: Japan 1987- (Fuzzy boom), Korea**
  Home electronics
  Vehicle control
  Process control
  Pattern recognition & image processing
  Expert systems
  Military systems (USA ~1990-)
  Space research
- **Applications to very complex control problems: Japan 1991-**
  E.G. helicopter autopilot

An application example

- One of the most interesting applications of fuzzy computing: “FOREX” system.
- 1989-1992, Laboratory for International Fuzzy Engineering Research (Yokohama, Japan) (Engineering – Financial Engineering)
- To predict the change of exchange rates (FOReign EXchange)
- ~5600 rules like:
  “IF the USA achieved military successes on the past day [E.G. in the Gulf War] THEN ¥/$ will slightly rise.”



Inputs

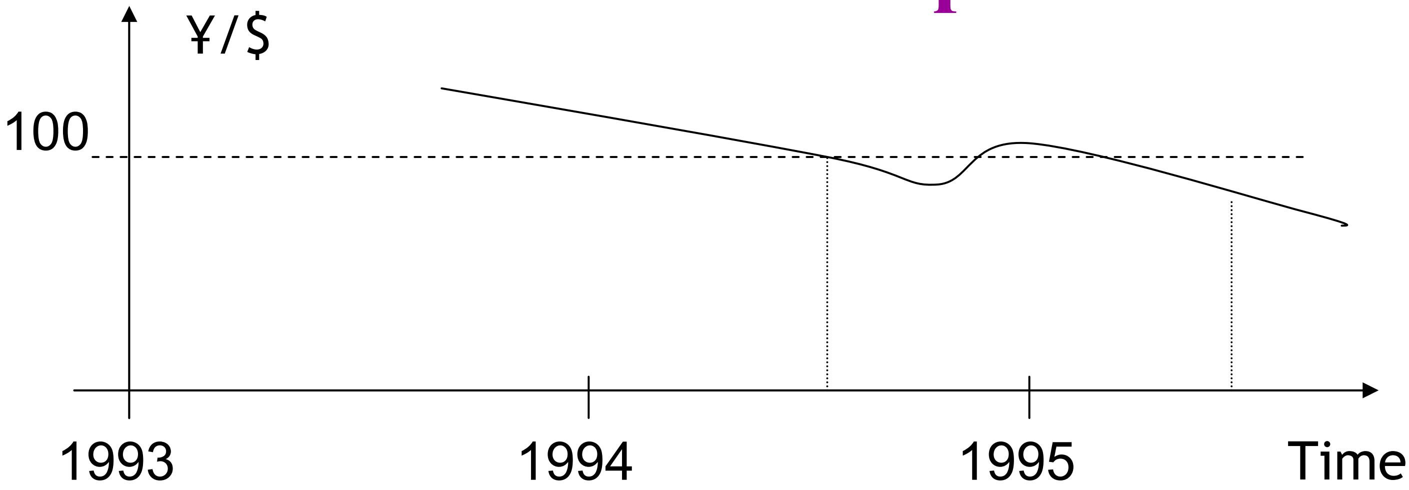
(Observations)

FOREX

Prediction

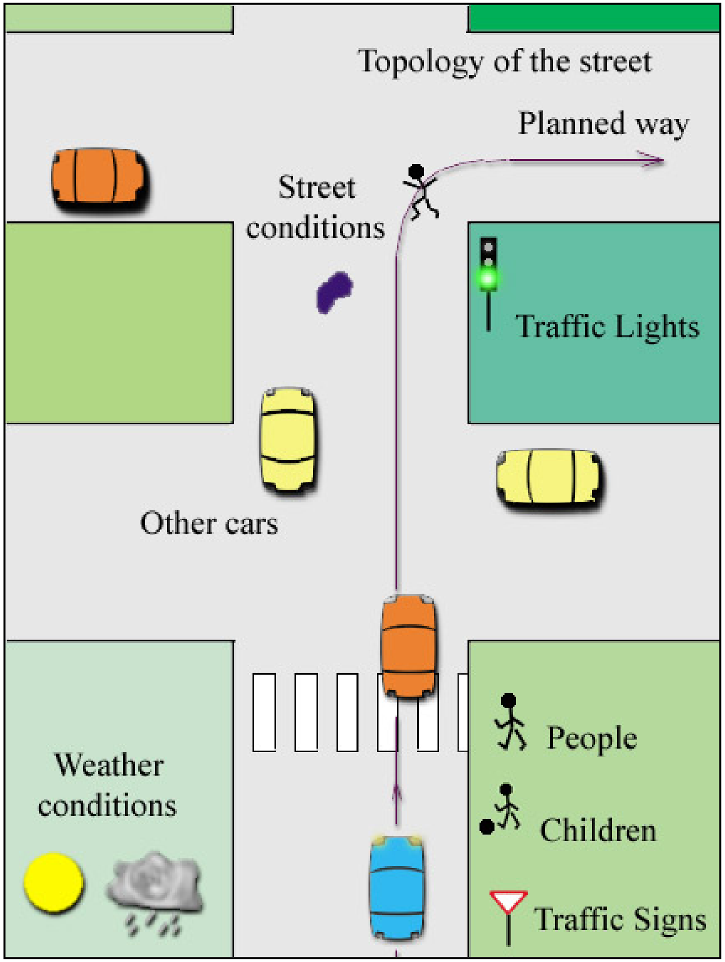
Fuzzy Inference Engine

Another Example



- What is fuzzy here?
- What is the tendency of the ¥/$ exchange rate?
  "It's MORE OR LESS falling" (The general tendency is "falling", there's no big interval of rising, etc.)
- What is the current rate?
  Approximately 88 ¥/$ → Fuzzy number
- When did it first cross the magic 100 ¥/$ rate? SOMEWHEN in mid 1995

A complex problem



Our car, save fuel, save time, etc.

- **Many components, very complex system.**
- **Can AI system solve it? Not, as far as we know.**
- **But WE can.**

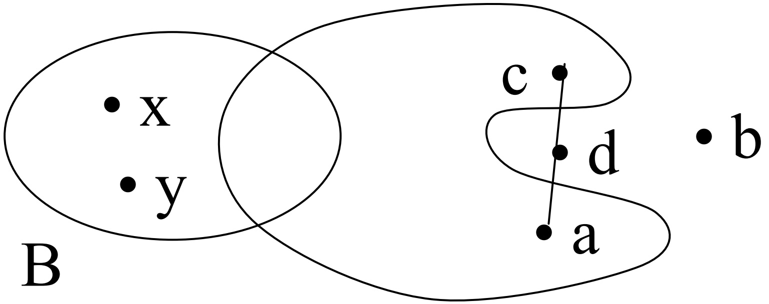
Definitions

- **Crisp set:**

$a \in A$

$b \notin A$

- **Convex set:**

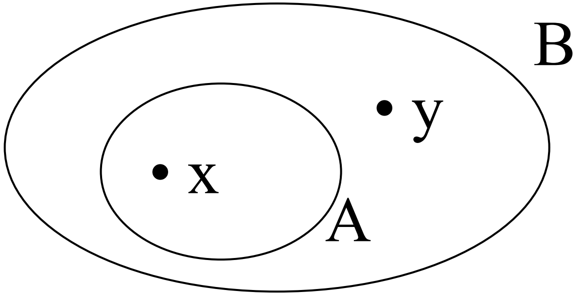


Crisp set A

A is not convex as $a \in A$, $c \in A$, but $d = \lambda a + (1-\lambda)c \notin A$, $\lambda \in [0, 1]$.

B is convex as for every $x, y \in B$ and $\lambda \in [0, 1]$ $z = \lambda x + (1-\lambda)y \in B$.

- **Subset:**



If $x \in A$ then also $x \in B$.

$A \subseteq B$

Definitions

- **Equal sets:**
  If $A \subseteq B$ and $B \subseteq A$ then $A = B$ if not so $A \neq B$.

- **Proper subset:**
  If there is at least one $y \in B$ such that $y \notin A$ then $A \subset B$.

- **Empty** set: No such $x \in \varnothing$.

- **Characteristic function:**
  $\mu A(x)$: $X \rightarrow \{0, 1\}$, where X the universe.
  0 value: x is not a member,
  1 value: x is a member.

Definitions

A={1, 2, 3, 4, 5, 6}

- **Cardinality: |A|=6.**
- **Power set of A:**

**P (A)={{}=Ø, {1}, {2}, {3}, {4}, {5}, {6},
{1, 2}, {1, 3}, {1, 4}, {1, 5}, {1, 6}, {2, 3}, {2, 4}, {2, 5}, {2, 6},
{3, 4}, {3, 5}, {3, 6}, {4, 5}, {4, 6}, {5, 6},
{1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {1, 2, 6}, {1, 3, 4}, {1, 3, 5}, {1, 3, 6}, {1, 4, 5},
{1, 4, 6}, {1, 5, 6}, {2, 3, 4}, {2, 3, 5}, {2, 3, 6}, {2, 4, 5}, {2, 4, 6}, {2, 5, 6},
{3, 4, 5}, {3, 4, 6}, {3, 5, 6}, {4, 5, 6},
{1, 2, 3, 4}, {1, 2, 3, 5}, {1, 2, 3, 6}, {1, 2, 4, 5}, {1, 2, 4, 6}, {1, 2, 5, 6},
{1, 3, 4, 5}, {1, 3, 4, 6}, {1, 3, 5, 6}, {1, 4, 5, 6}, {2, 3, 4, 5}, {2, 3, 4, 6},
{2, 3, 5, 6}, {2, 4, 5, 6}, {3, 4, 5, 6},
{1, 2, 3, 4, 5}, {1, 2, 3, 4, 6}, {1, 2, 4, 5, 6}, {1, 3, 4, 5, 6},
{2, 3, 4, 5, 6}, {1, 2, 3, 4, 5, 6}}.**

$|P\ (A)|=2^{|6|}=64$.

Definitions

- **Relative complement** or **difference**:
  A–B={x | x∈A and x∉B}
  A={1, 2, 3, 4, 5, 6}, B={1, 3, 4, 5}, A–B={2, 6}.
  C={1, 3, 4, 5, 7, 8}, A–C={2, 6}!
- **Complement**: $\overline{A} = X - A$ where X is the universe.
  Complementation is **involutive**: $\overline{\overline{A}} = A$
  Basic properties: $\overline{\varnothing} = X,$

  $$\overline{X} = \varnothing$$
- **Union**:
  A∪B={x | x∈A or x∈B}
  For $\{A_i \mid i \in I\}$ $\bigcup_{i=1}^{n} A_i = \{x \mid x \in A_i \text{ for some } i\}$

$$A \cup X = X$$

$$A \cup \varnothing = A$$

$$A \cup \overline{A} = X$$

(Law of excluded middle *("a kizárt harmadik")*)

Definitions

- **Intersection:**

  $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

For $\{A_i \mid i \in I\}$ $\bigcap_{i=1}^{n} A_i = \{x \mid x \in A_i \text{ for all } i\}$

$A \cap \varnothing = \varnothing$

$A \cap X = A$

$A \cap \overline{A} = \varnothing$ (Law of contradiction)

More properties:

Commutativity: $A \cup B = B \cup A$, $A \cap B = B \cap A$.

Associativity: $A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$,
$A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C)$.

Idempotence: $A \cup A = A$, $A \cap A = A$.

Distributivity: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Definitions

- **More properties (continued):**

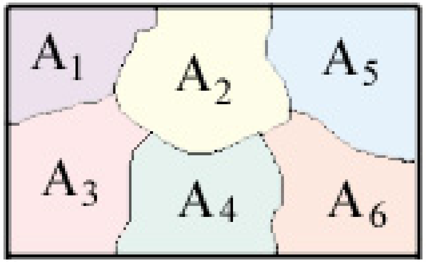
  DeMorgan's laws:

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

- **Disjoint sets: $A \cap B = \varnothing$.**
- **Partition of X:**

$$\Pi(x) = \left\{ A_i \mid i \in I,\, A_{i_1} \cap A_{i_2} = \varnothing,\, \bigcup_{i \in I} A_i = X \right\}$$



$$X = \bigcup_{i=1}^{6} A_i$$

$$A_i \cap A_j = \varnothing$$

$$\{A_i \mid i \in N_6\} = \Pi(x)$$

Summarize properties

Involution	$\overline{\overline{A}} = A$
Commutativity	$A \cup B = B \cup A$, $A \cap B = B \cap A$
Associativity	$A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$, $A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Idempotence	$A \cup A = A$, $A \cap A = A$
Absorption	$A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$
Absorption of complement	$A \cup \left(\overline{A} \cap B\right) = A \cup B$ $A \cap \left(\overline{A} \cup B\right) = A \cap B$
Absorption by X and ∅	$A \cup X = X$, $A \cap \varnothing = \varnothing$
Identity	$A \cup \varnothing = A$, $A \cap X = A$
Law of contradiction	$A \cap \overline{A} = \varnothing$
Law of excluded middle	$A \cup \overline{A} = X$
DeMorgan's laws	$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\qquad$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Membership function

Crisp set

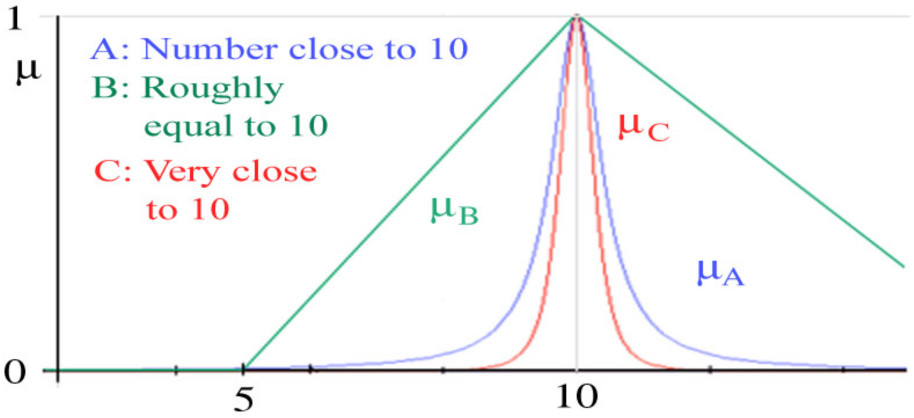
Characteristic function

$\mu_A : X \rightarrow \{0, 1\}$

Fuzzy set

Membership function

$\mu_A : X \rightarrow [0, 1]$



$$\mu_A = \frac{1}{1+5(x-10)^2}$$

$$\mu_B = \begin{cases} 0 & x \le 5 \vee x > 17 \\ 0.2(x-5) & 5 < x \le 10 \\ -\frac{1}{7}(x-17) & 10 < x \le 17 \end{cases}$$

$$\mu_C = \mu_A^2$$

Fuzzy Sets

- Formal definition:

  A fuzzy set A in X is expressed as a set of ordered pairs:

$$A = \{(x, \underbrace{\mu_A}(x)) | x \in X\}$$

Fuzzy set

Membership function (MF)

Universe or universe of discourse

- A fuzzy set is totally characterized by a membership function (MF)

Fuzzy Sets with Discrete Universes

- Fuzzy set C = “desirable city to live in”

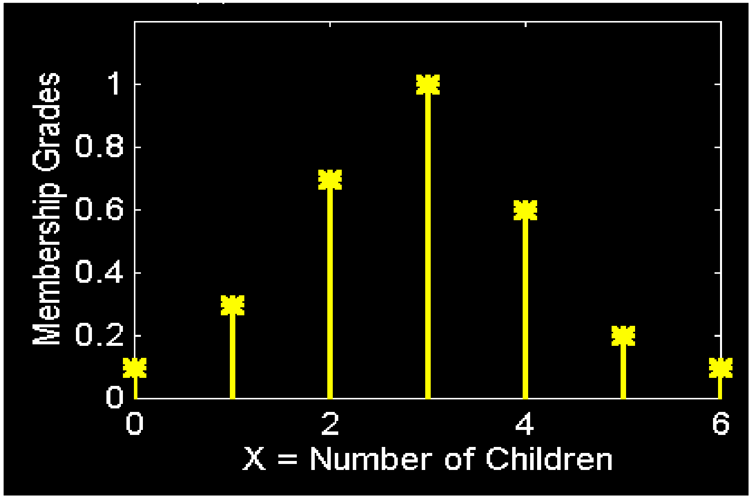
  X = {SF, Boston, LA} (discrete and nonordered)

  C = {(SF, 0.9), (Boston, 0.8), (LA, 0.6)}

- Fuzzy set A = “sensible number of children”

  X = {0, 1, 2, 3, 4, 5, 6} (discrete universe)

  A = {(0, .1), (1, .3), (2, .7), (3, 1), (4, .6), (5, .2), (6, .1)}



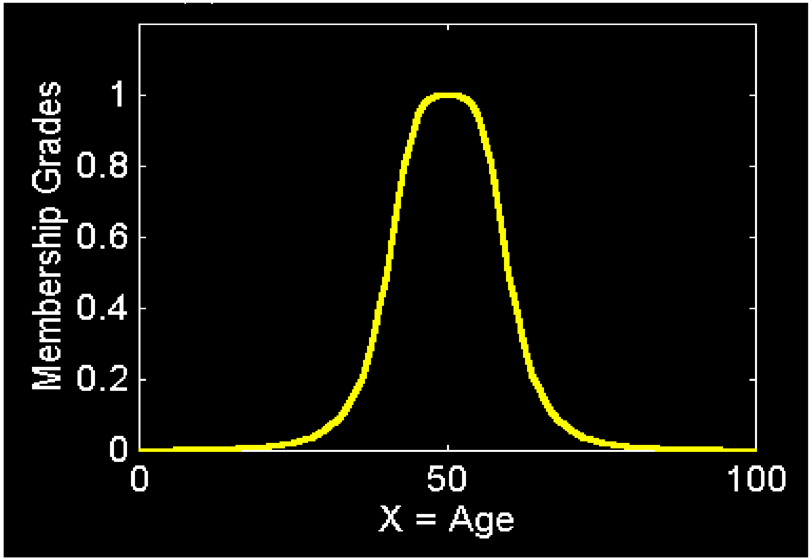
Fuzzy Sets with Continuous Universes

- **Fuzzy set B = “about 50 years old”**

  X = Set of positive real numbers (continuous)

  B = {(x, μ_B(x)) | x in X}

$$\mu_B(x) = \frac{1}{1 + \left(\frac{x-50}{10}\right)^2}$$



Alternative Notation

- **A fuzzy set A can be alternatively denoted as follows:**
  - **X is discrete:**

$$A = \sum_{x_i \in X} \mu_A(x_i) / x_i$$

  - **X is continuous:**

$$A = \int_X \mu_A(x) / x$$

- **Note that Σ and integral signs stand for the union of membership grades; "/" stands for a marker and does not imply division.**

Fuzzy Sets - example

- **Triangular MF**

$$trimf(x;a,b,c)=\max\left(\min\left(\frac{x-a}{b-a},\frac{c-x}{c-b}\right),0\right)$$

- **Trapezoidal MF**

$$trapmf(x;a,b,c,d)=\max\left(\min\left(\frac{x-a}{b-a},1,\frac{d-x}{d-c}\right),\right.$$

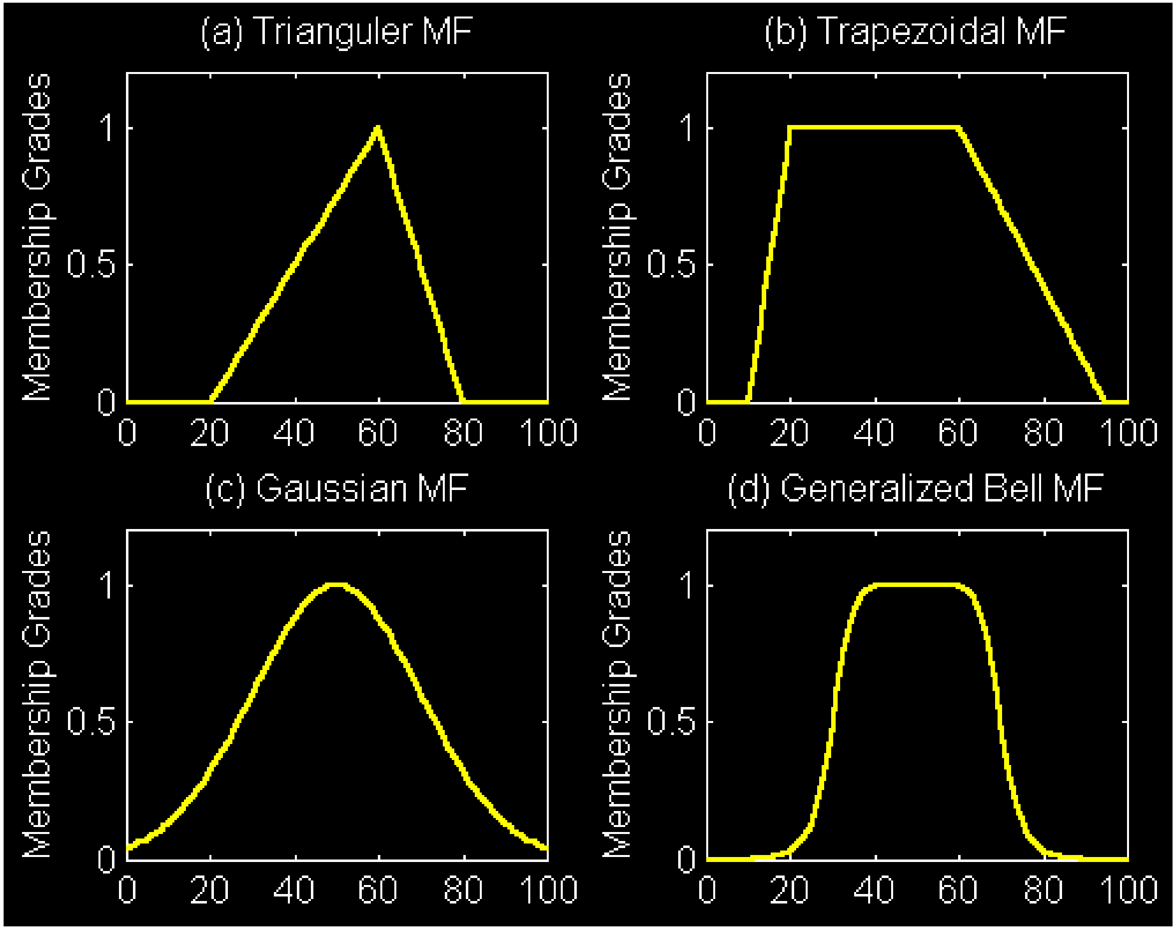
- **Gaussian MF**

$$gaussmf(x;a,b,c)=e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^{2}}$$

- **Generalized bell MF**

$$gbellmf(x;a,b,c)=\frac{1}{1+\left|\frac{x-c}{b}\right|^{2b}}$$

Fuzzy Sets - example

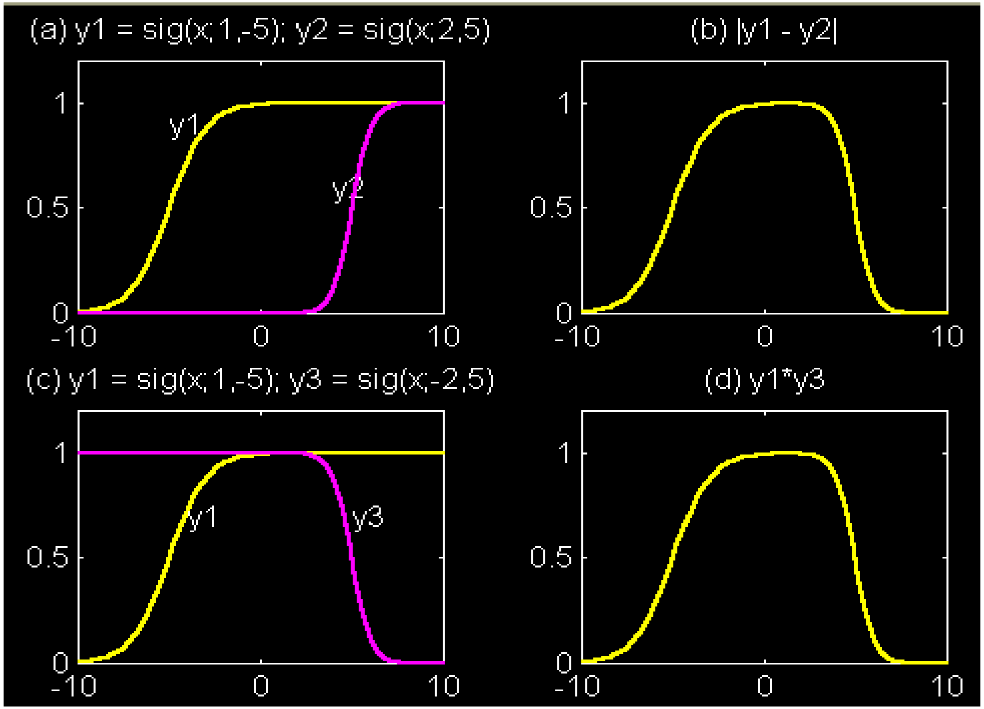


Fuzzy Sets - example

- **Sigmoidal MF**

$$sigmf\,(x;a,b,c) = \frac{1}{1+e^{-a(x-c)}}$$

- **Extensions**
  - **Absolute difference of two sig. MF**
  - **Product of two sig. MF**



Fuzzy Sets - example

- **L-R MF**

$$LR(x;c,\alpha,\beta)=\begin{cases}F_L\left(\dfrac{c-x}{\alpha}\right), x<c\\ F_R\left(\dfrac{x-c}{\beta}\right), x\geq c\end{cases}$$

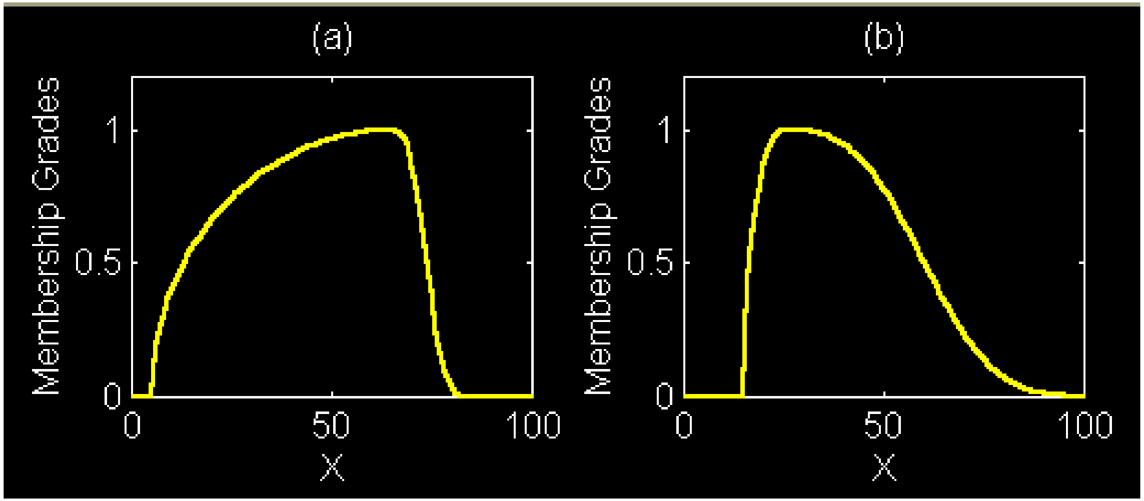
$$F_L(x)=\sqrt{\max(0,1-x^2)}\quad F_R(x)=\exp(-|x|^3)$$

- **Example**

c=65

a=60

b=10



c=25

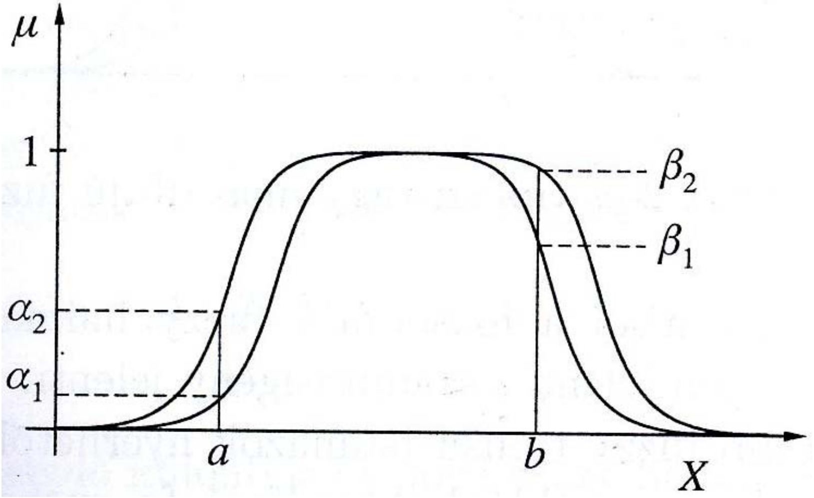
a=10

b=40

Membership function

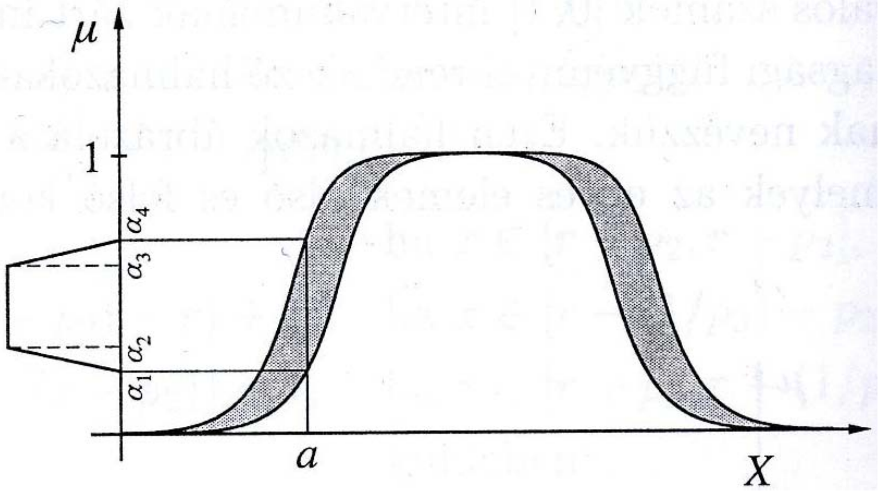
Interval valued fuzzy set:

$$\mu : X \to \varepsilon([0,1])$$



Fuzzy valued fuzzy set:

$$\mu : X \to \widetilde{\mathrm{P}}([0,1])$$



More general idea:

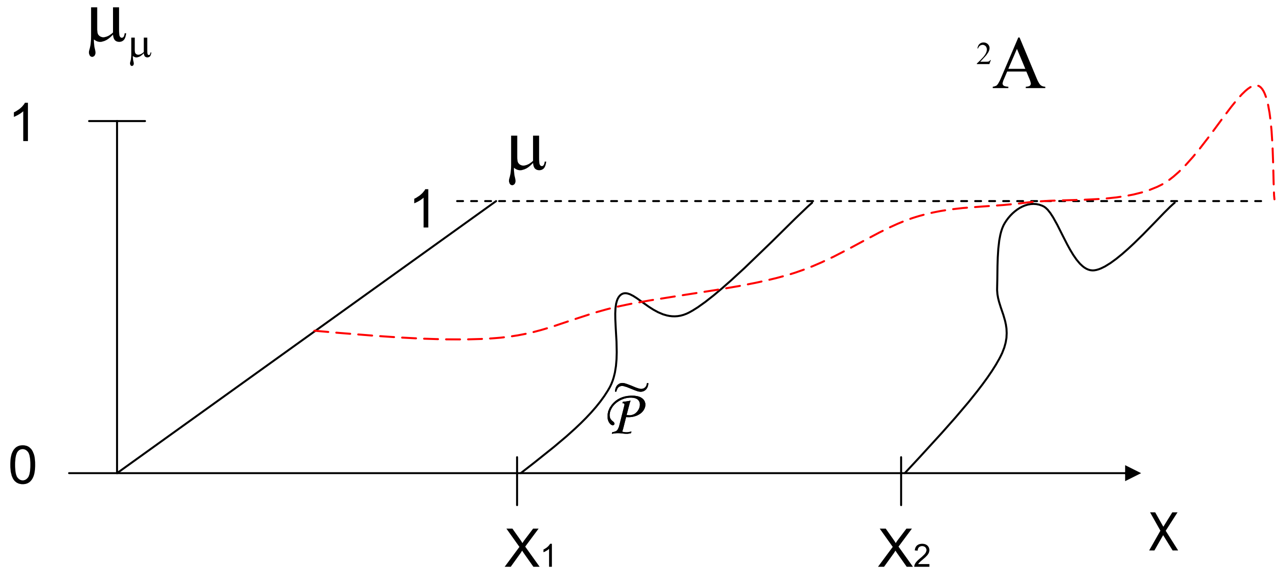
L-Fuzzy set, μ_A: X→L, L has a linear partial (or full) ordering.

(L: Lattice)

Definitions

- **Fuzzy valued fuzzy set: Fuzzy set of type 2**

$$\mu : X \to \tilde{\mathrm{P}}([0,1])$$



- **Fuzzy set of type N**

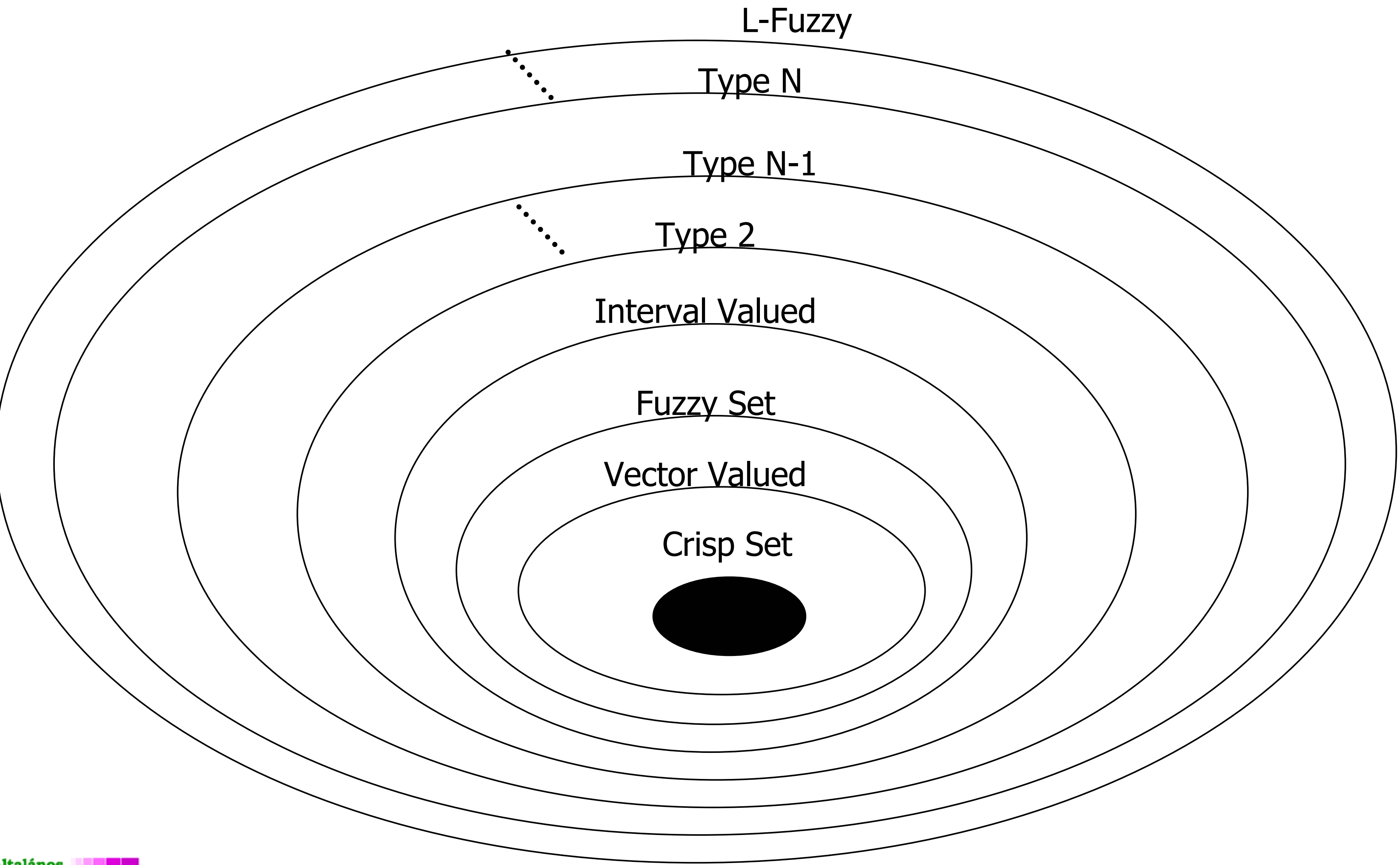
$$\mu : \tilde{\mathcal{P}}^{k\text{-}1}(x) \to [0,1]$$

$$\tilde{\mathcal{P}}^{k}(x) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}^{k\text{-}1}(x)) \quad k > 1$$

$$\tilde{\mathcal{P}}^{0}(x) = x$$

Definitions

- **Relation of various classes of sets discussed**



Membership function

Fuzzy measure:

g: P (x)→[0, 1]

Degree of belonging to a crisp subset of the universe.

Example:

’teenager’=13..19 years old,

‘twen’=20..29 years old,
‘in his thirties’=30..39 years old, etc.

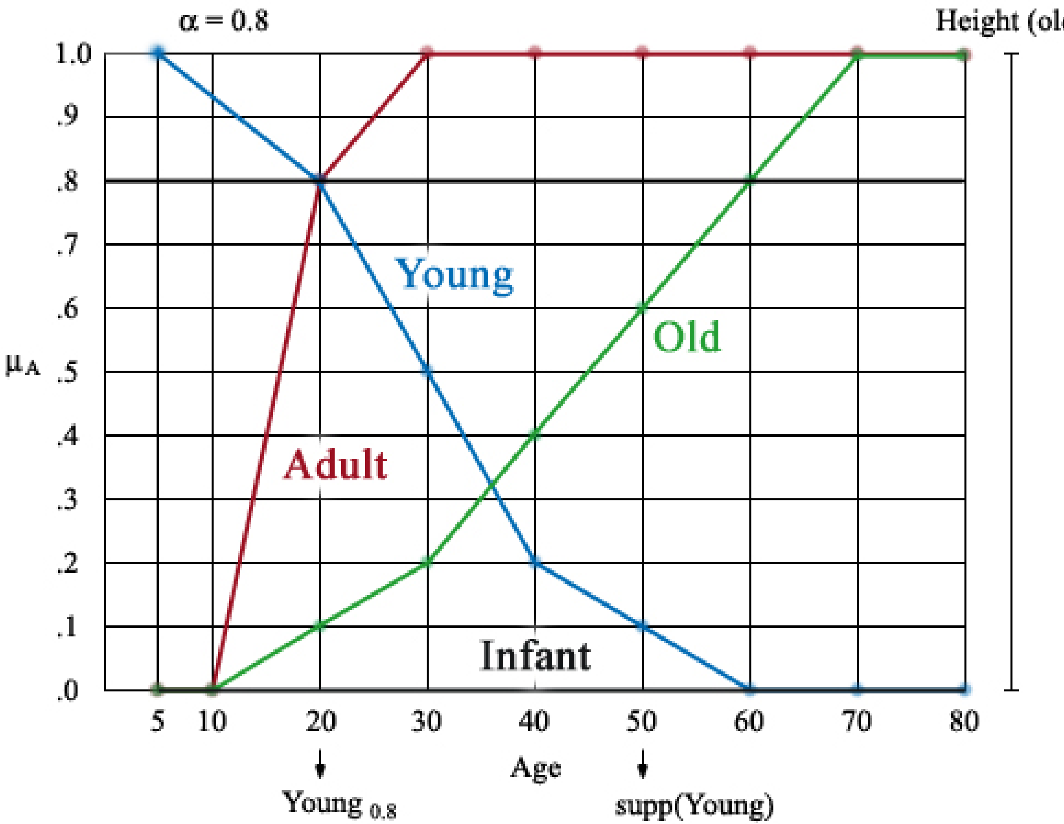
X={Years of men’s possible age}
‘teenager’⊂X

g(‘Susan is a twen’)=0.8,
g(‘Susan is a teenager’)=0.2,
g(‘Susan is in her thirties’)=0.1.

Best guess **has highest measure.**

Some basic concepts of fuzzy sets

Elements	Infant	Adult	Young	Old
5	0	0	1	0
10	0	0	1	0
20	0	.8	.8	.1
30	0	1	.5	.2
40	0	1	.2	.4
50	0	1	.1	.6
60	0	1	0	.8
70	0	1	0	1
80	0	1	0	1



Some basic concepts of fuzzy sets

- **Support**: **supp(A)={x | μ_A(x)>0}.**

  supp: $\tilde{\mathcal{P}}(x) \rightarrow \mathcal{P}(x)$

  μ_{Infant}=0, so
  supp(Infant)=0

- **If |supp(A)|<∞, A can be defined**

  $$A=\mu_1/x_1+\mu_2/x_2+\ldots+\mu_n/x_n.$$

  $$A=\sum_{i=1}^{n}\mu_i/x_i \quad A=\int_X \mu_A(x)/x$$

- **Kernel (Nucleus, Core):**

  Kernel(A)={x | μ_A(x)=1}.

Definitions

Height:

$$\text{Height}(A) = \max_{x}(\mu_A(x)) \Rightarrow \sup_{x}(\mu_A(x))$$

- **If height(A)=1 A is normal**
- **If height(A)<1 A is subnormal**
- **height(0)=0**
- **(If height(A)=0 then supp(A)=0)**

height(Old)=1
height(Infant)=0

α-cut:

$$A_\alpha = \{x \mid \mu_A(x) \geq \alpha\}$$

Strong α- Cut:

$$A_{\bar{\alpha}} = \{x \mid \mu_A(x) > \alpha\}$$

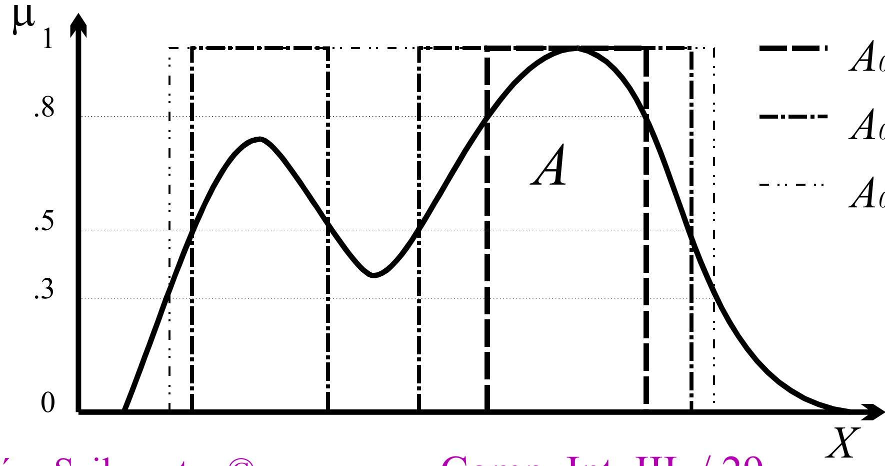
$$\text{Young}_{0.8} = \{5, 10, 20\}$$

$$\text{Young}_{\overline{0.8}} = \{5, 10\}$$

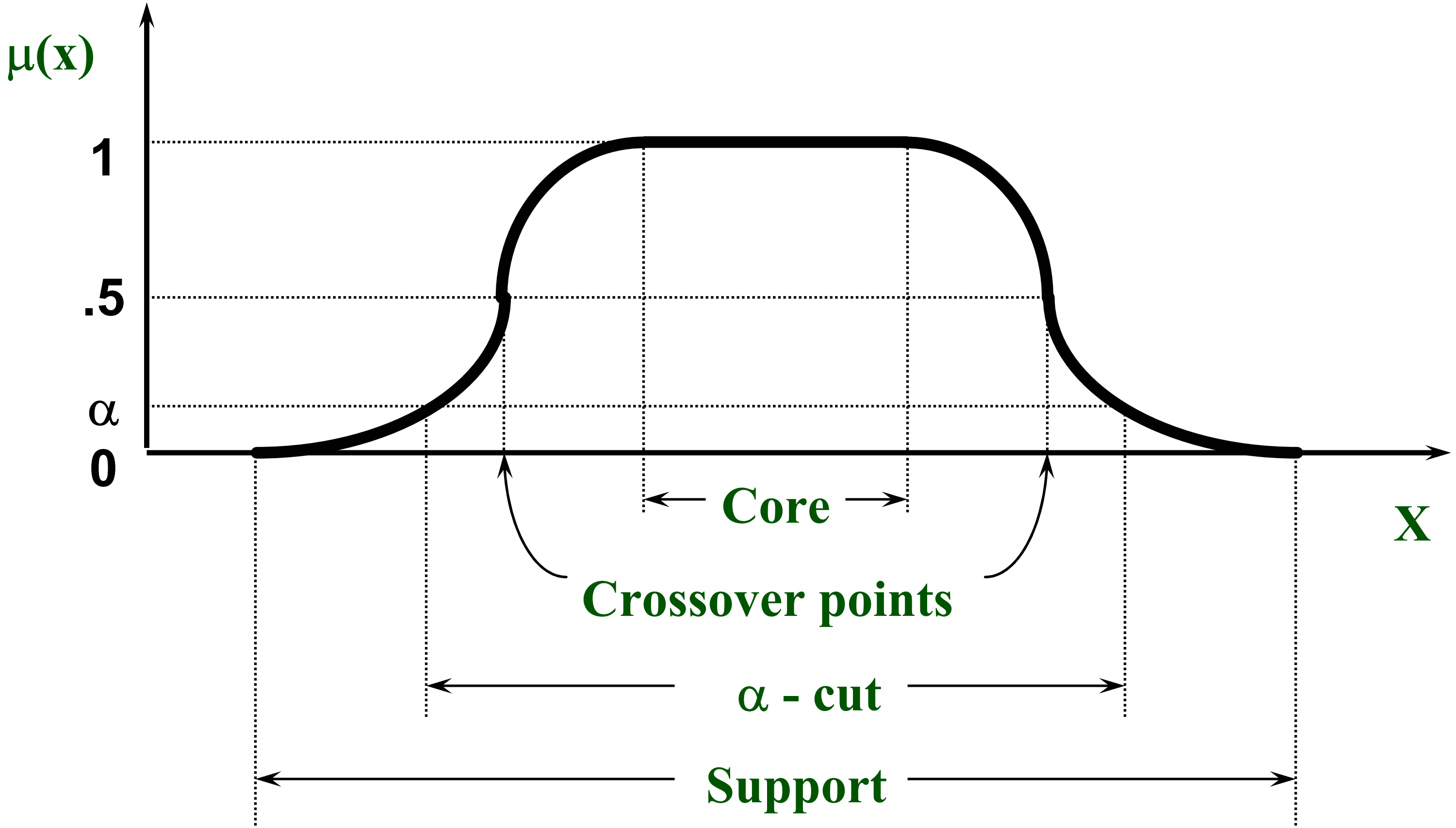
- **Kernel:** $A_1 = \{x \mid \mu_A(x) = 1\}$
- **Support:** $A_{\bar{0}} = \{x \mid \mu_A(x) > 0\}$

- **If A is subnormal, Kernel(A)=0**

$$A_\alpha \leq A_\beta \quad \text{IF} \quad \alpha \geq \beta$$



Definitions



Definitions

- **Level set** of *A*: Set of all levels α ∈ [0,1] that represent distinct α-cuts of a given fuzzy set *A*

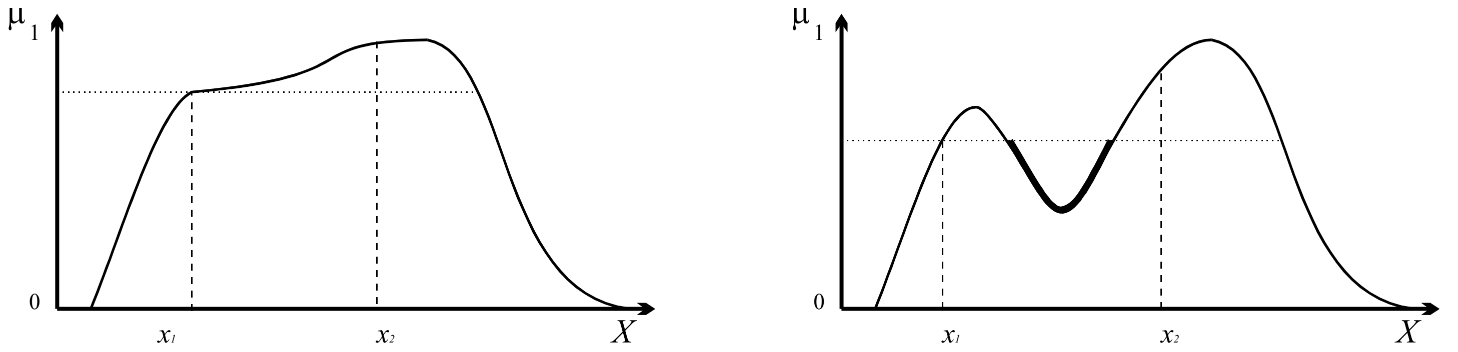
$$\Lambda_A = \{\alpha \mid \mu_A(x) = \alpha \ \ for \ \ some \ \ x \in X\}$$

- **Convex fuzzy set:**

$$\mathrm{X} = \Re^{\mathrm{n}}$$

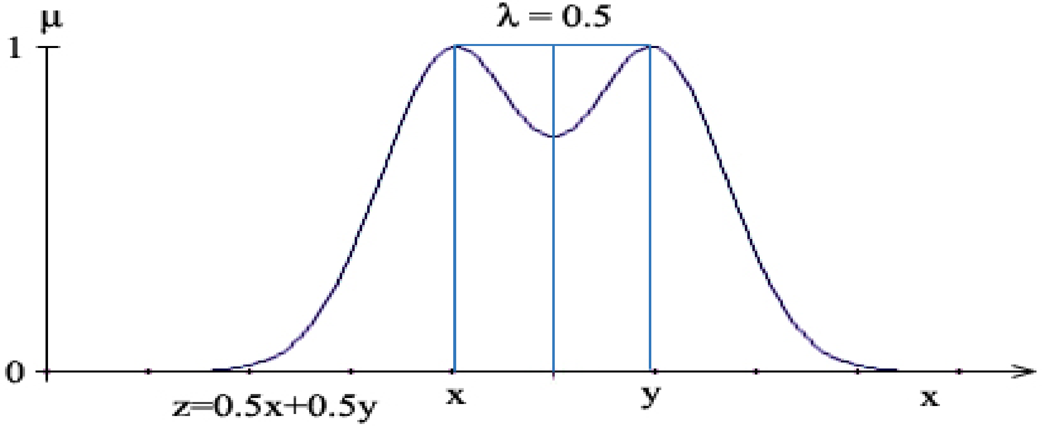
  - **A is convex if for every $x_1, x_2 \in X$ and $\lambda \in [0,1]$:**

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

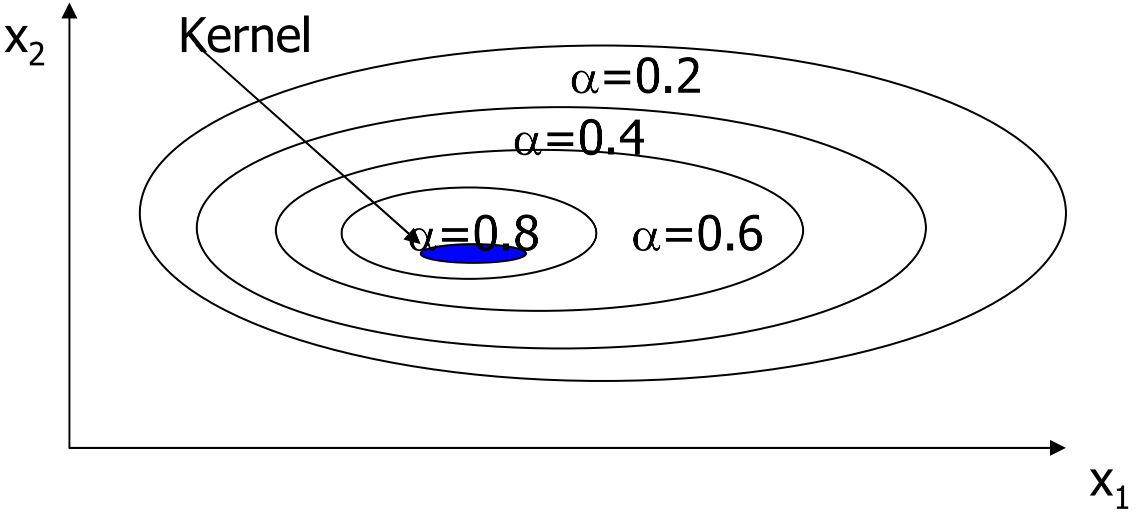


Definitions

- **Nonconvex fuzzy set:**



- **Convex fuzzy set over R ^{2}**

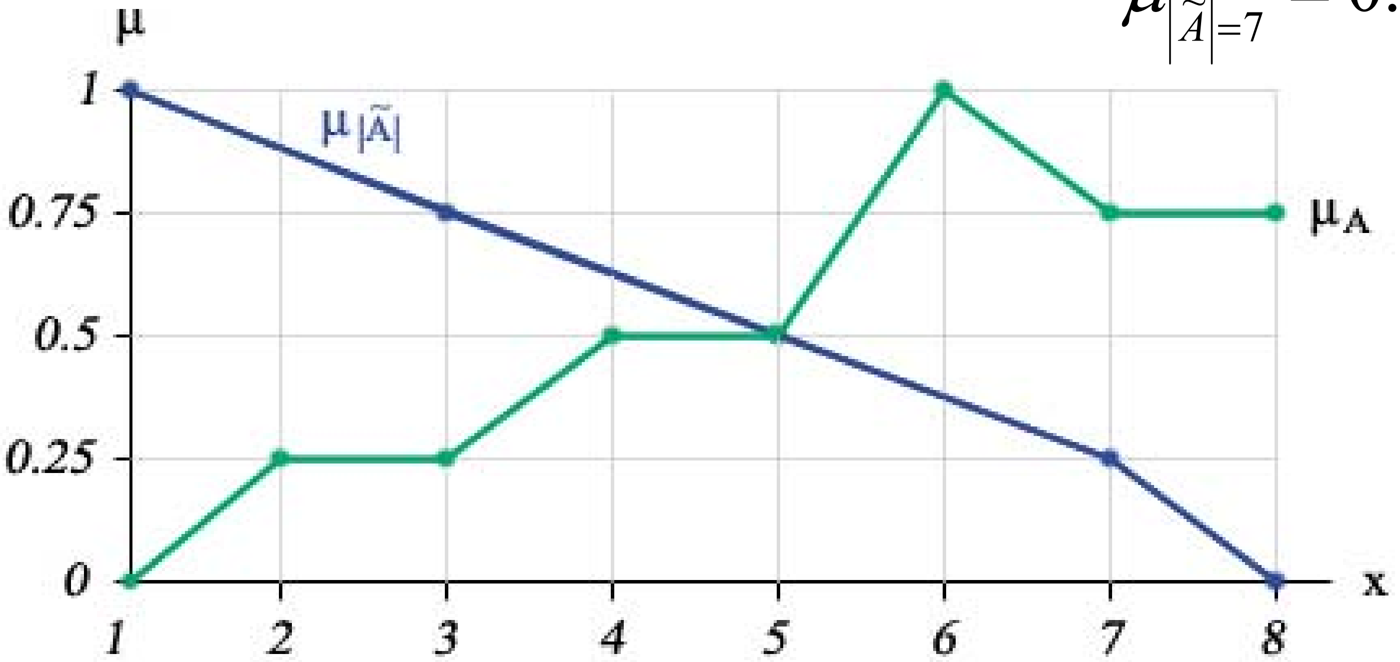


Definitions

- **Fuzzy cardinality of FS A:** $\left|\tilde{A}\right|$

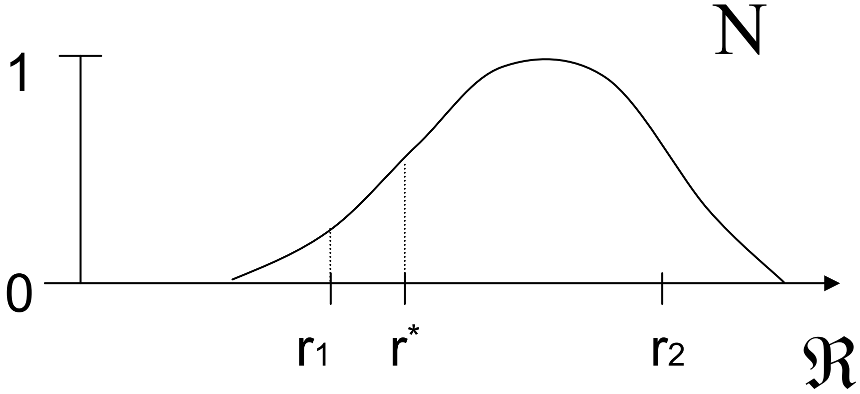
$$\mu_{\left|\tilde{A}\right|}\left(\left|A_\alpha\right|\right)=\alpha \quad \textbf{for all } \alpha\in\Lambda_A$$

$$\mu_{\left|\tilde{A}\right|=7}=0.25$$



Definitions

- **Fuzzy number**: **Convex and normal fuzzy set of $\Re$**
  - Example 1:

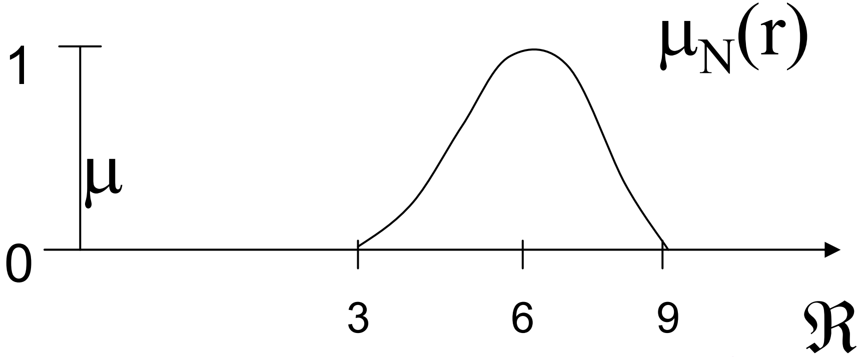


$\text{height}(\mu_N(r))=1$

for any r_1, r_2: $\mu_N(r^*)=\mu_N(\lambda r_1+(1-\lambda)r_2) \geq \min(\mu_N(r_1), \mu_N(r_2))$

  - Example 2: "Approximately equal to 6"

$$\mu_N(r)=\begin{cases}1-\sqrt{\dfrac{|r-6|}{3}} & \text{if} \quad r\in[3, 9]\\ 0 & \text{otherwise}\end{cases}$$



Definitions

- **Flat fuzzy number:**

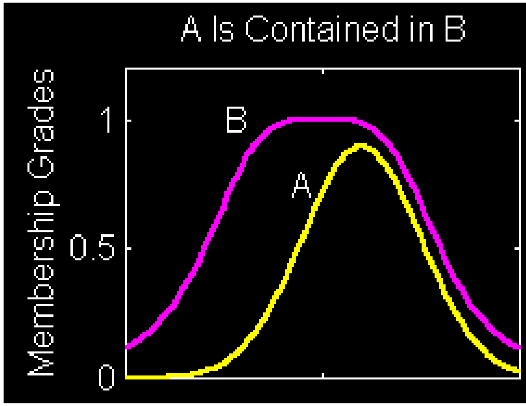
  There is a,b ($a \neq b$, $a,b \in \Re$) $\mu_N(r)=1$ IFF $r \in [a,b]$

  (Extension of 'interval')

- **Containment (inclusion) of fuzzy set**

  $A \subseteq B$ IFF $\mu_A(x) \leq \mu_B(x)$

  - **Example: Old $\subseteq$Adult**



- **Equal fuzzy sets**

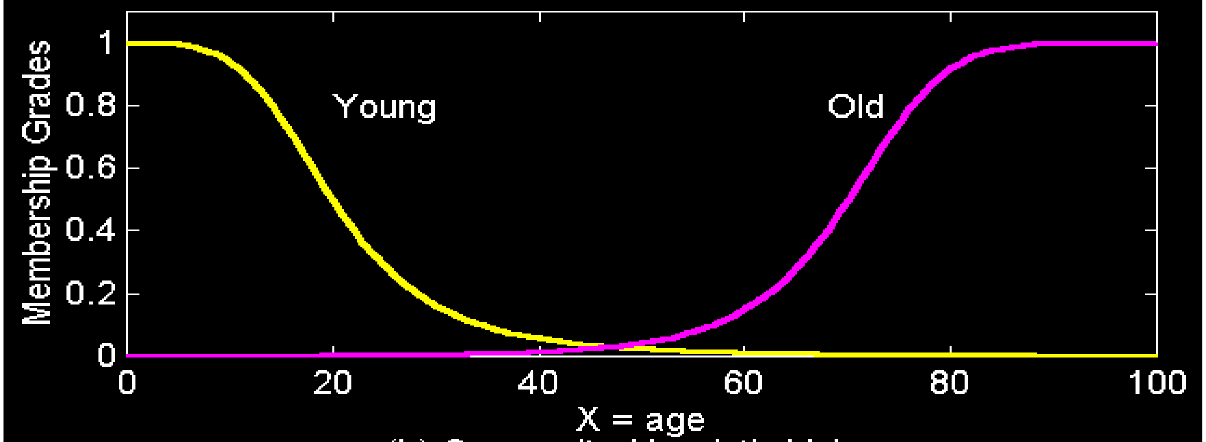
  $A=B$ IFF $A \subseteq B$ and $A \supseteq B$ If it is not the case: $A \neq B$

- **Proper subset**

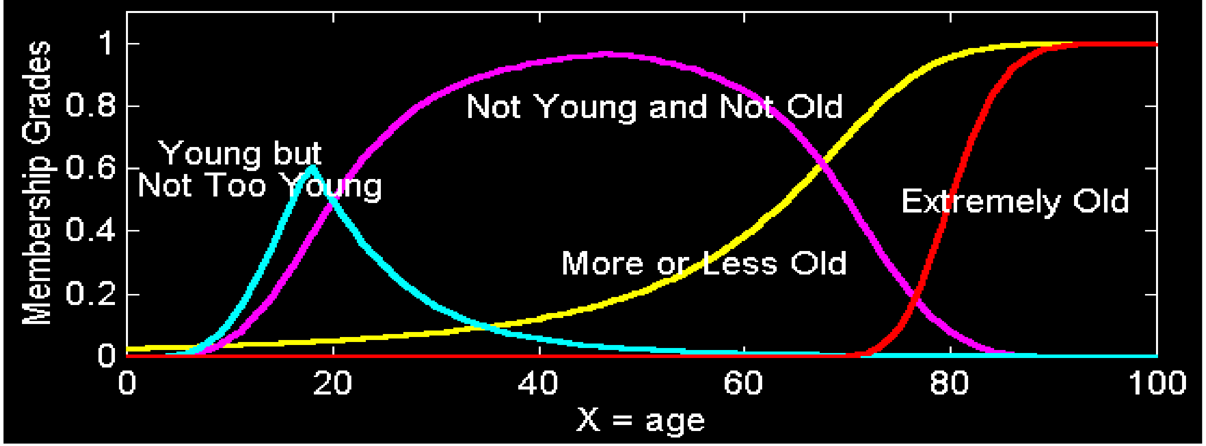
  $A \subseteq B$ IFF $A \subseteq B$ and $A \neq B$

Linguistic Values (Terms)

(a) Primary Linguistic Values

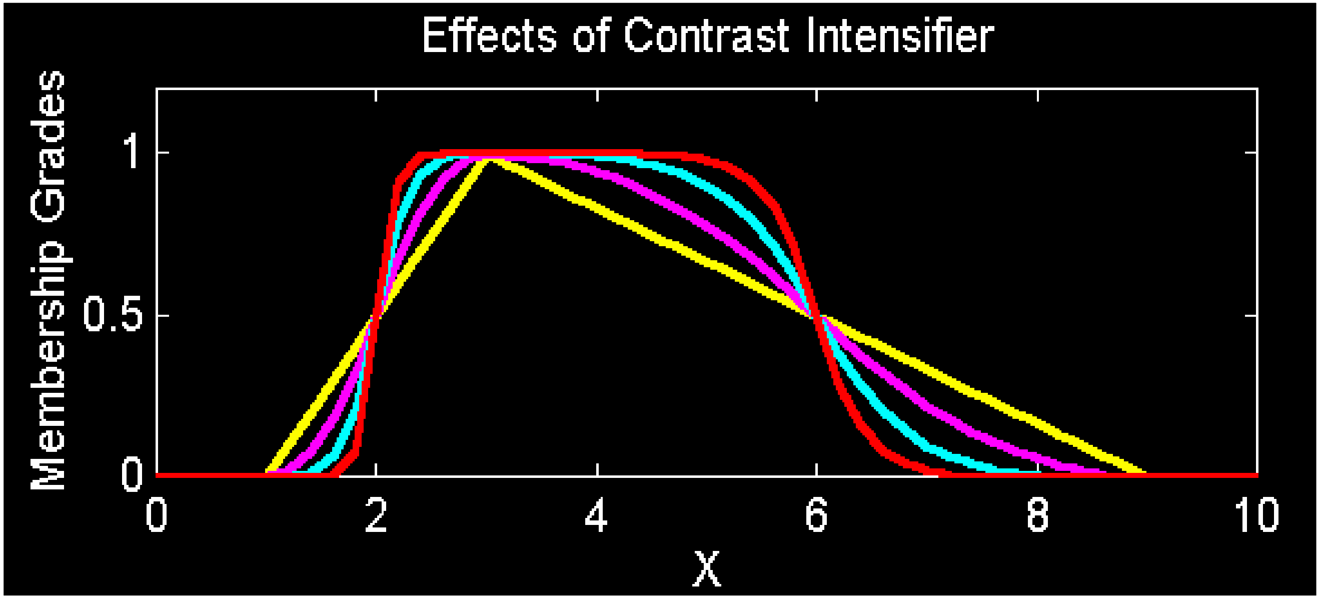


(b) Composite Linguistic Values



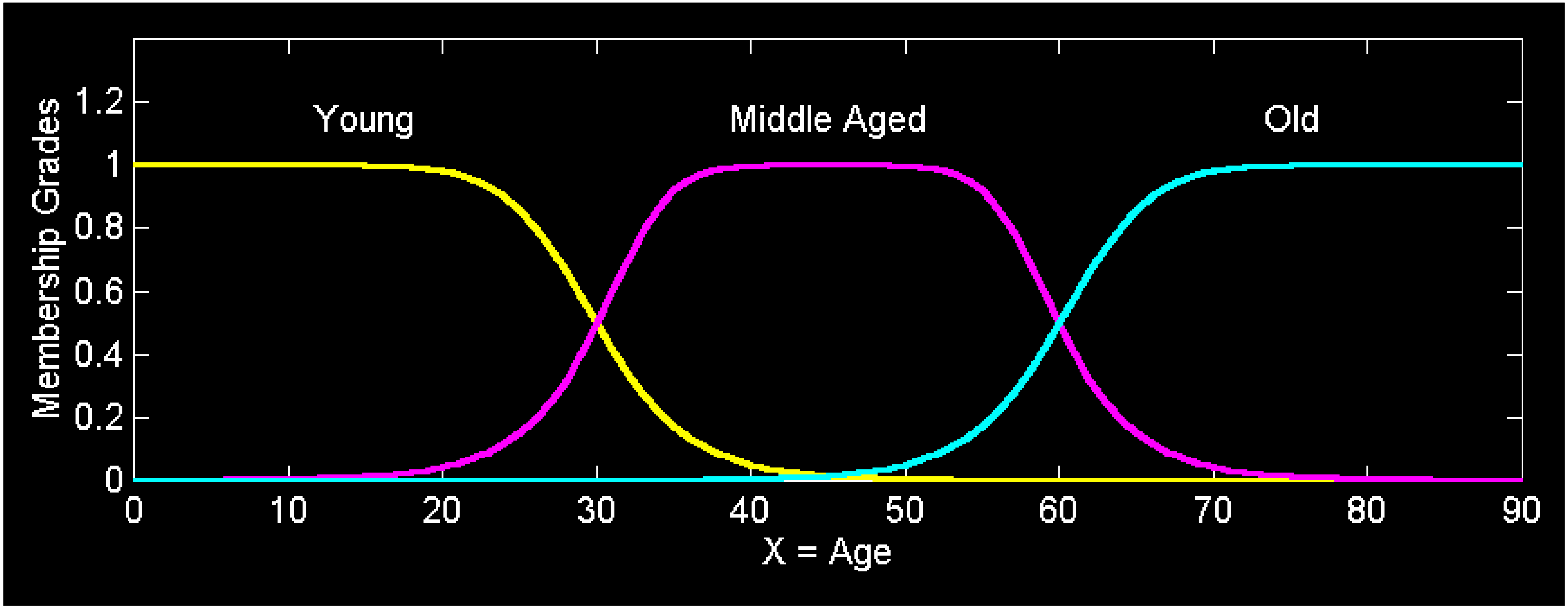
Operations (hedges) on Linguistic Values – e.g.

- **Concentration:** $$CON(A) = A^2$$
- **Dilation:** $$DIL(A) = A^{0.5}$$
- **Contrast intensification:** $$INT(A) = \begin{cases} 2A^2, & 0 \le \mu_A(x) \le 0.5 \\ \neg 2(\neg A)^2, & 0.5 \le \mu_A(x) \le 1 \end{cases}$$



Fuzzy Partition

- **Fuzzy partitions formed by the linguistic values “young”, “middle aged”, and “old”:**



Definitions - Extension principle

- A general method for extending nonfuzzy mathematical concepts to deal with fuzzy quantities

•*A* is a fuzzy set on *X* :

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_n)/x_n$$

•The image of *A* under *f*() is a fuzzy set *B*:

$$B = \mu_B(x_1)/y_1 + \mu_B(x_2)/y_2 + \cdots + \mu_B(x_n)/y_n$$

•where $y_i = f(x_i)$, $i = 1$ to n.

•If *f*() is a many-to-one mapping, then

$$\mu_B(y) = \max_{x=f^{-1}(y)} \mu_A(x) \qquad \mu_B(y) = 0 \text{ if } f^{-1}(y) = \varnothing$$

$$\mu(y_i) = \max\left(\mu(x_j) \mid f(x_j) = y_i\right)$$

Definitions - Extension principle – e.g.

- **Arithmetics with fuzzy numbers: Using extension principle**

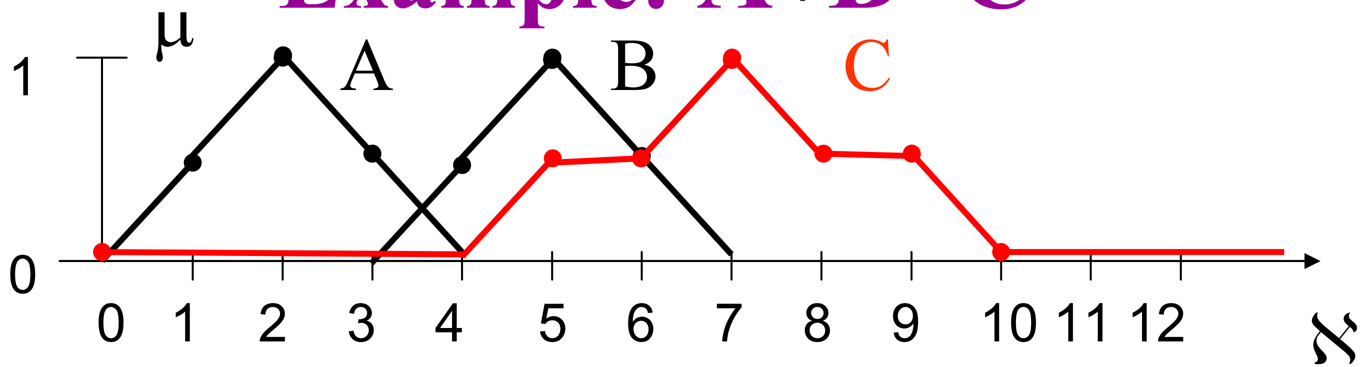
$$a+b=c \qquad (a,b,c \in \aleph)$$

$$A \tilde{+} B = C$$

$$\mu_C(n) = \max_{a,b\in\aleph} \min\left(\mu_A(a), \mu_B(b) \mid a+b=n\right)$$

⟹

Example: A+B=C



$$\mu_C(n) = \max_{a,b\in\aleph} \min\left(\mu_A(a), \mu_B(b) \mid a+b=n\right)$$

n	a	b	min(μ_A,μ_B)
3	0	3	0
4	0 1	4 3	0 0
5	0 1 2	5 4 3	0 0.5 0
6	0 1 2 3	6 5 4 3	0 0.5 0.5
7	0 1 …	7 6	0 0.5

n	a	b	min(μ_A,μ_B)
7	2 3 4	5 4 3	1 0.5 0
8	1 2 3 4	7 6 5 4	0 0.5 0.5 0
9	2 3 4	7 6 5	0 0.5 0
10	3 4	7 6	0 0
11	4	7	0

Definitions

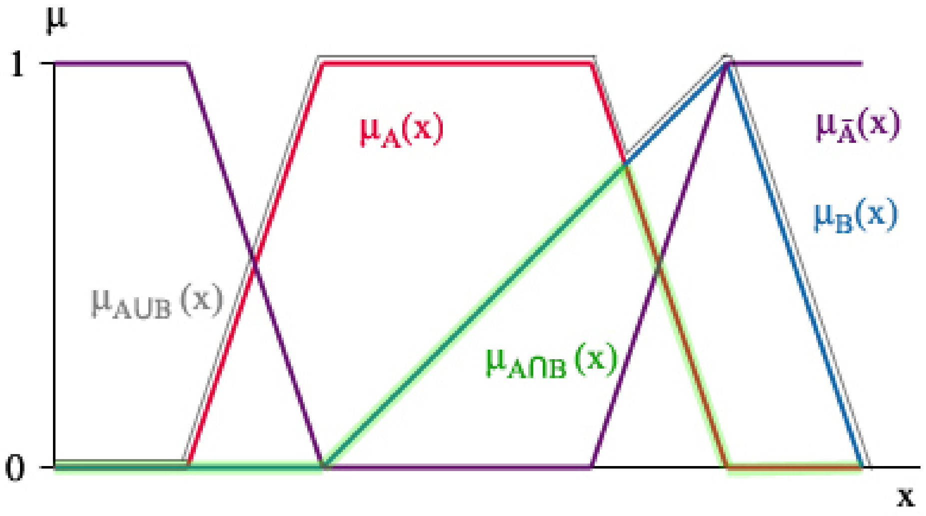
Fuzzy set operations defined by L.A. Zadeh in 1964/1965

- **Complement:**
- **Intersection:**
- **Union:**

$$\mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

$$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

$$\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$$

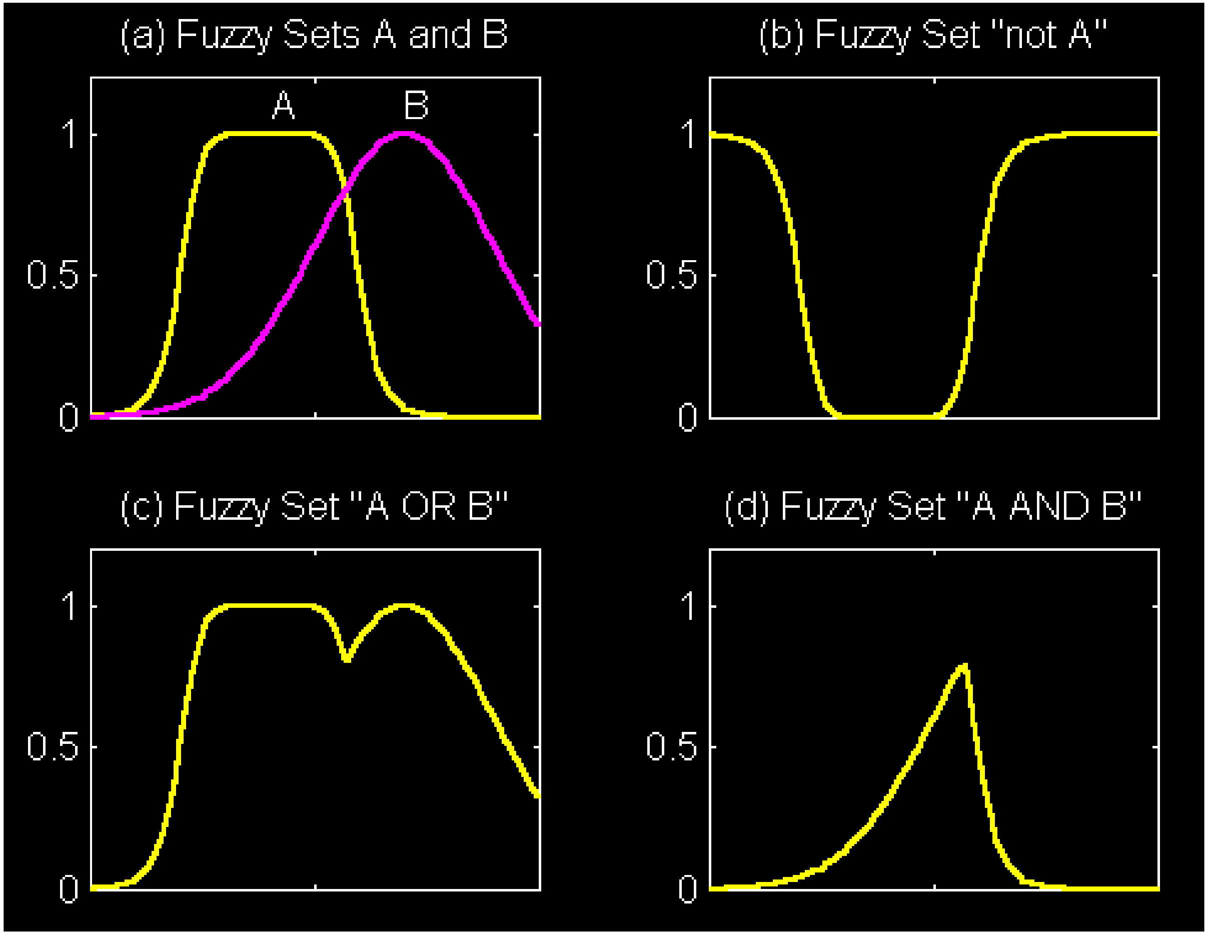


Definitions

Fuzzy set operations defined by L.A. Zadeh in 1964/1965

$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$ $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$

$\mu_{\overline{A}}(x) = 1 - \mu_A(x)$



Definitions

This is really a generalization of crisp set op's!

A	B	¬A	A∩B	A∪B	$1-\mu_A$	min	max
0	0	1	0	0	1	0	0
0	1	1	0	1	1	0	1
1	0	0	0	1	0	0	1
1	1	0	1	1	0	1	1

$$\mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

$$\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

$$\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$$

Classical (Two valued) logic

- **Classical (two valued logic):**
  Any logic –
  Means of a reasoning propositions: true (1) or false (0).
  Propositional logic uses logic variables, every variable stands for a proposition.
  When combining logic variables new variables (new propositions) can be constricted.
  Logical function:
  f: $v_1, v_2, \ldots, v_n \rightarrow v_{n+1}$
- 2^{2^n} **different logic functions of n variables exist.**
  E.g. if n=2, 16 different logic functions (see next page).

Logic functions of two variables

v_2	1100	Adopted name	Symbol	Other names used
v_1	1010			
w_1	0000	Zero fn.	0	Falsum
w_2	0001	Nor fn.	$v_1 \downarrow v_2$	Pierce fn.
w_3	0010	Inhibition	$v_1 > v_2$	Proper inequality
w_4	0011	Negation	$\neg v_2$	Complement
w_5	0100	Inhibition	$v_1 < v_2$	Proper inequality
w_6	0101	Negation	$\neg v_1$	Complement
w_7	0110	Exclusive or function	$v_1 \oplus v_2$	Antivalence
w_8	0111	Nand function	$v_1 \uparrow v_2$	Sheffer Stroke
w_9	1000	Conjunction	$v_1 \wedge v_2$	And function
w_{10}	1001	Biconditional	$v_1 \otimes v_2$	Equivalence
w_{11}	1010	Assertion	v_1	Identity
w_{12}	1011	Implication	$v_1 \Leftarrow v_2$	Conditional, inequality
w_{13}	1100	Assertion	v_2	Identity
w_{14}	1101	Implication	$v_1 \Rightarrow v_2$	Conditional, inequality
w_{15}	1110	Disjunction	$v_1 \vee v_2$	Or function
w_{16}	1111	One fn.	1	Verum

Definitions

- **Important concern:**
  How to express all logic functions by a few logic primitives (functions of one or two logic variables)?

  A system of logic primitives is (functionally) complete if all logic functions can be expressed by the functions in the system.

  A system is a base system if it is functionally complete and when omitting any of its elements the new system isn't functionally complete.

Definitions

- A system of logic functions is functionally complete if:
  - At least one doesn't preserve 0
  - At least one doesn't preserve 1
  - At least one isn't monotonic
  - At least one isn't self dual
  - At least one isn't linear
- **Example:** AND , NOT

$$\overline{A} \qquad \overline{0} \neq 0$$

$$\overline{A} \qquad \overline{1} \neq 1$$

$$\overline{A} \qquad \overline{0} = 1, \quad \overline{1} = 0, \quad 0 < 1, \quad 1 \geq 0$$

$$A \cap B \qquad \overline{\overline{A} \cap \overline{B}} \neq A \cap B \ \left(= A \cup B\right)$$

$$A \cap B \quad CANNOT \quad BE \quad EXPRESSED$$

$$BY \quad ONLY \quad \oplus$$

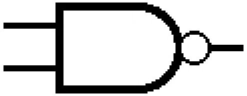
Definitions

Importance of base systems, and functionally complete systems.

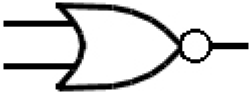
Digital engineering:

Which set of primitive digital circuits is suitable to construct an arbitrary circuit?

- **Very usual: AND, OR, NOT (Not easy from the point of view of technology!)**
- **NAND (Sheffer stroke)**



- **NOR (Pierce function)**



- **NOT, IMPLICATION (Very popular among logicians.)**

Definitions

- **Logic formulas (LF):**
  - **E.g.: Let's adopt +,-,* as a complete system. Then:**
    - 0 and 1 are LFs
    - If v is a LF $\overline{v}$ is a LF
    - If A and B are LFs, then A+B, A*B are LFs
    - There are no other LFs
  - **Similarly with -, →, etc.**
- There are **infinitely many ways to describe a logic formula** in an equivalent way
  - **E.g.: A, $\overline{\overline{A}}$, A+A, A*A, A+A+A, etc.**
- **Canonical formulas, normal form**
  - **DCF** $\overline{A}B + AB + A\overline{B}$ $(= A + B)$ *(Disjunctive)*
  - **CCF** $(\overline{A} + \overline{B})(A + B)$ $(= \overline{A}B + A\overline{B})$ *(Conjunctive)*
- **Logic formulas with identical truth values are named equivalent**

Definitions

- **Always true logic formulas:**
  Tautology: $A \leftrightarrow A + A$
- **Always false logic formulas:**
  Contradiction: $A \cdot \overline{A}$
- **Various forms of tautologies are used for didactive inference = *inference rules***

$$a \rightarrow b = \overline{a} + b = \overline{a \cdot \overline{b}}$$

  - **Modus Ponens**:
    $(a * (a \rightarrow b)) \rightarrow b$
  - **Modus Tollens**:
    $(\overline{b} * (a \rightarrow b)) \rightarrow \overline{a}$
  - **Hypothetical Syllogism** (Rezolúció):
    $((a \rightarrow b) * (b \rightarrow c)) \rightarrow (a \rightarrow c)$

$$\frac{\alpha \vee \beta,\ \neg\beta \vee \gamma}{\alpha \vee \gamma}$$

$$\frac{\neg\,\alpha \longrightarrow \beta,\ \beta \longrightarrow}{\neg\,\alpha \longrightarrow \gamma}$$

  - **Rule of Substitution**:
    If in a tautology any variable is replaced by a logic formula then it remains a tautology.

Definitions

- These inference rules form the base of expert systems and related systems (even fuzzy control!)
- Abstract algebraic model:
  Boolean Algebra

  $$\mathbf{B}=(B,+,*,-)$$

  B has at least two different elements (bounds): 0 and 1 + some properties of binary operators "+" and "*", and unary operator "-".

Properties of boolean algebras

B1. Idempotence	$a+a=a$, $a\cdot a=a$
B2. Commutativity	$a+b=b+a$
B3. Associativity	$(a+b)+c=a+(b+c)$, $(a\cdot b)\cdot c=a\cdot(b\cdot c)$
B4. Absorption	$a+(a\cdot b)=a$, $a\cdot(a+b)=a$
B5. Distributivity	$a\cdot(b+c)=(a\cdot b)+(a\cdot c)$, $a+(b\cdot c)=(a+b)\cdot(a+c)$
B6. Universal bounds	$a+0=a$, $a+1=1$ $a\cdot 1=a$, $a\cdot 0=0$
B7. Complementarity	$a+\neg a=1$, $a\cdot\neg a=0$, $\neg 1=0$
B8. Involution	$\neg(a+b)= \neg a \cdot \neg b$
B9.Dualization	$\neg(a\cdot b)= \neg a + \neg b$

Lattice

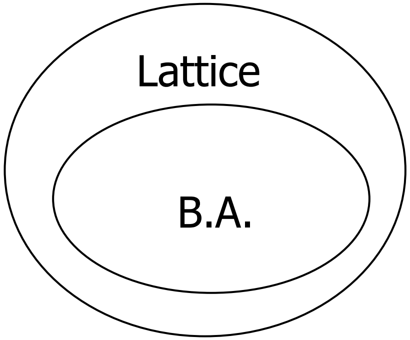
Definitions

Set theory	Boolean algebra	Propositional logic
P(X)	B	F(V)
$\cup$	$+$	$\vee$
$\cap$	$*$	$\wedge$
-	-	-
X	1	1
$\varnothing$	0	0
$\subseteq$	$\leq$	$\Rightarrow$

- **Correspondences defining isomorphisms between set theory, boolean algebra and propositional logic**

Definitions

- **Isomorphic structure of crisp set and logic operations = <u>boolean</u> algebra**



- **Structure of propositions: x is P**
  - $\underbrace{\text{Dr. Kóczy}}_{\substack{x_1 \\ SUBJECT}}$ is $\underbrace{\text{above 190cm}}_{\substack{P \\ PREDICATE}} = TRUE$
  - **x_2=Dr.Kim** **$P(x_2)$=FALSE!**

Definitions

- **Quantifiers** (egzisztenciális, univerzális kvantorok)**:**
  - **$(\exists x)$ P(x): There exists an x such that x is P**
  - **$(\forall x)$ P(x): For all x, x is P**
  - **$(\exists ! x)$ P(x): $\exists$x and only one x such that x is P**

Definitions

- **Two valued logic questioned since B.C.**
  Three valued logic includes indeterminate value: ½
  Negation: 1-a, ∧∨⇒⇔ differ in these logics.
- ***Examples:***

ab	Łukasiewicz ∧∨⇒⇔	Bochvar ∧∨⇒⇔	Kleene ∧∨⇒⇔	Heyting ∧∨⇒⇔	Reichenbach ∧∨⇒⇔
00	0011	0011	0011	0011	0011
0½	0½1½	½½½½	0½1½	0½10	0½1½
01	0110	0110	0110	0110	0110
½0	0½½½	½½½½	0½½½	0½00	0½½½
½½	½½11	½½½½	½½½½	½½11	½½11
½1	½11½	½½½½	½11½	½11½	½11½
10	0100	0100	0100	0100	0100
1½	½1½½	½½½½	½1½½	½1½½	½1½½
11	1111	1111	1111	1111	1111

Definitions

- No difference from classical logic for 0 and 1.

But: $a \cdot \overline{a} = 0, a + \overline{a} = 1$ **are not true!**

(excluded middle)

- **Quasi tautology: doesn't assume 1.**
  Quasi contradiction: doesn't assume 0.

- Next step?

N-valued logic

- **N-valued logic:**

$$T_n = \left\{0, \frac{1}{n-1}, \; \dots \; , \frac{n-2}{n-1}, 1\right\}$$

Degrees of truth

(Lukasiewicz, ~1933)

$$\begin{aligned}
&\overline{a} = 1 - a\\
&a \wedge b = \min(a, b)\\
&a \vee b = \max(a, b)\\
&a \rightarrow b = \min(1, 1 + b - a)\\
&a \leftrightarrow b = 1 - |a - b|
\end{aligned}$$

LOGIC

PRIMITIVES

Definitions

- L_n n=2, … , ∞

  ↑ (ξ_0)

  Classical Logic

← Rational truth values

If T_∞ is extended to [0,1] we obtain (T_{ξ_1}) L_1 with continuum truth degrees

- L_1 is isomorphic with fuzzy set

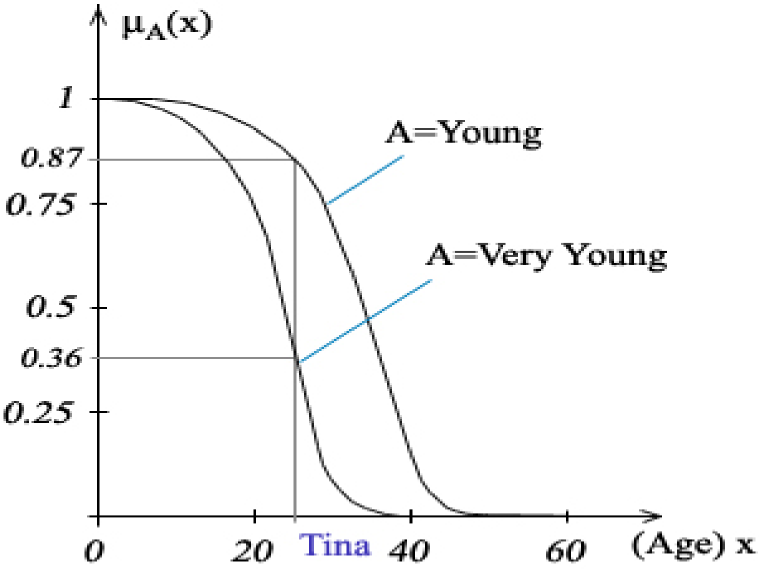
It is enough to study one of them, it will revelal all the facts above the other.

- Fuzzy logic must be the foundator of approximate reasoning, based on natural language!

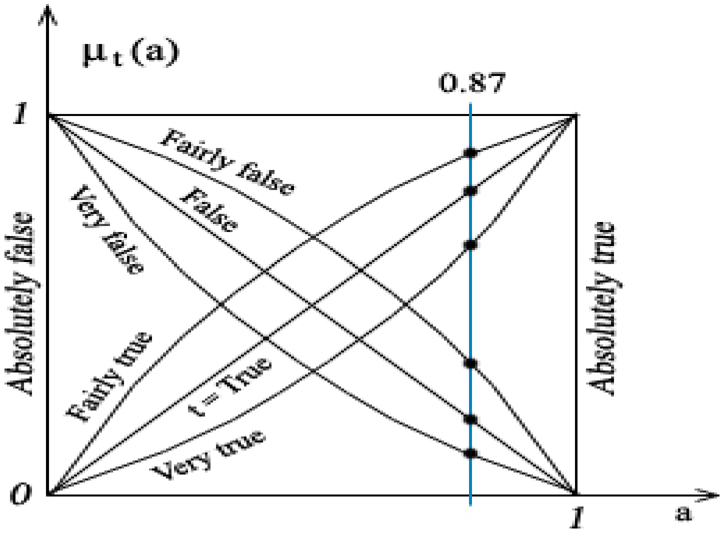
Fuzzy Proportion

- **Fuzzy proportion: X is P**
  'Tina is young', where:
  'Tina': crisp age, 'young': fuzzy predicate.

Fuzzy sets expressing linguistic terms for ages



Truth claims – Fuzzy sets over [0, 1]



- **Fuzzy logic based approximate reasoning is most important for applications!**

Fuzzy Operations - Complement

- **Usually** $\mu_A(x)\in(0,1)$ **and** $\mu_{\bar{A}}(x)\in(0,1)$

$$c:[0,1]\rightarrow[0,1]$$

$$\mu_{\bar{A}}(x)=c(\mu_A(x))$$

- **Axioms (skeleton):**

c1 $c(0)=1 \quad \wedge \quad c(1)=0$ **(boundary conditions)**

c2 $\forall a,b\in[0,1] \quad a<b\Rightarrow c(a)\geq c(b)$ **(monotonicity)**

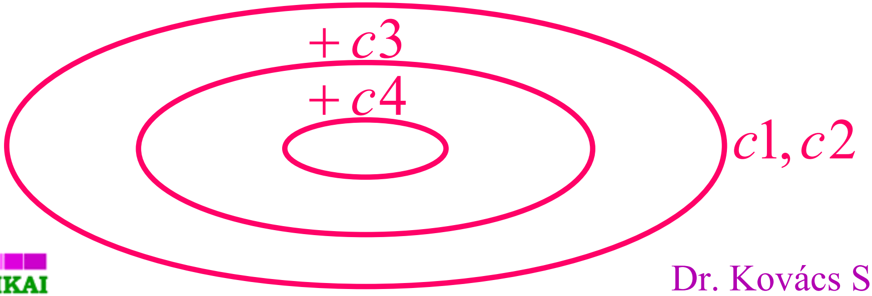
- **A family of functions C satisfy c1,c2**

$$C:\tilde{P}(x)\rightarrow\tilde{P}(x) \qquad c(\mu_A(x))=\mu_{C(A)}$$

- **Practical additions to the axioms:**

c3. c is a continuous function

c4. c is involutive $(c[c(a)]=a, \quad \forall a\in[0,1])$



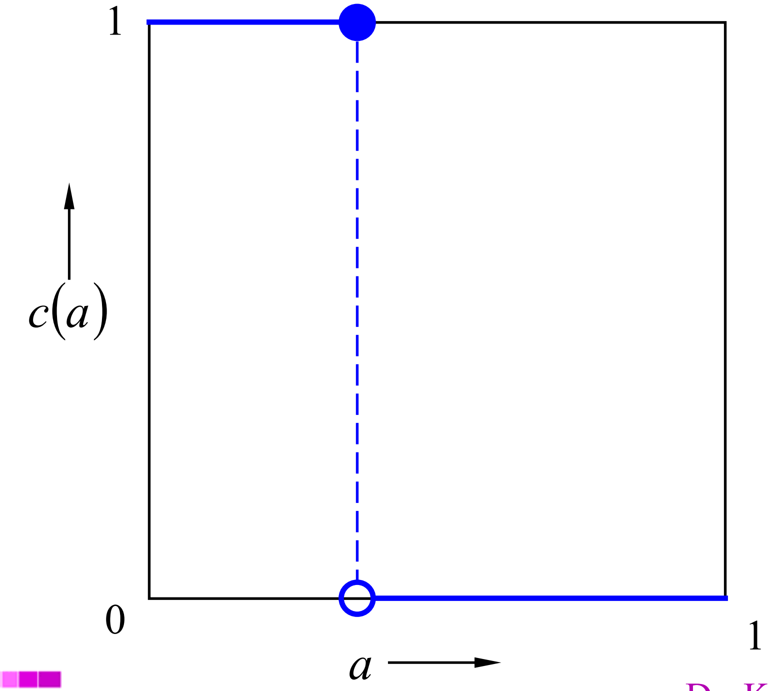
Fuzzy Operations – Complement – e.g.

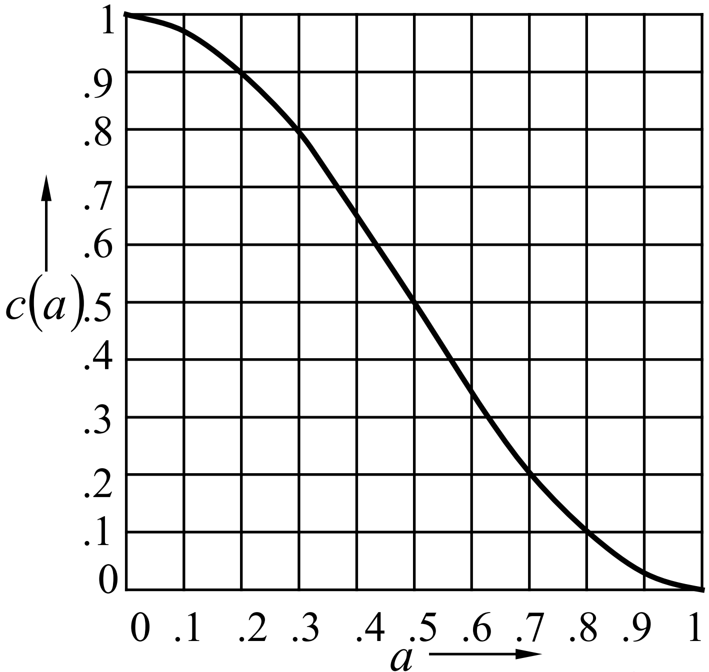
Some examples

- **1.,** $$c(a)=\begin{cases}1 & for \quad a\le t \quad a\in[0,1]\\ 0 & for \quad a>t \quad t\in[0,1)\end{cases}$$ **satisfies c1, c2**

  t = threshold

- **2.,** $$c(a)=\frac{1}{2}(1+\cos a\pi)$$ **satisfies c1, c2, c3**

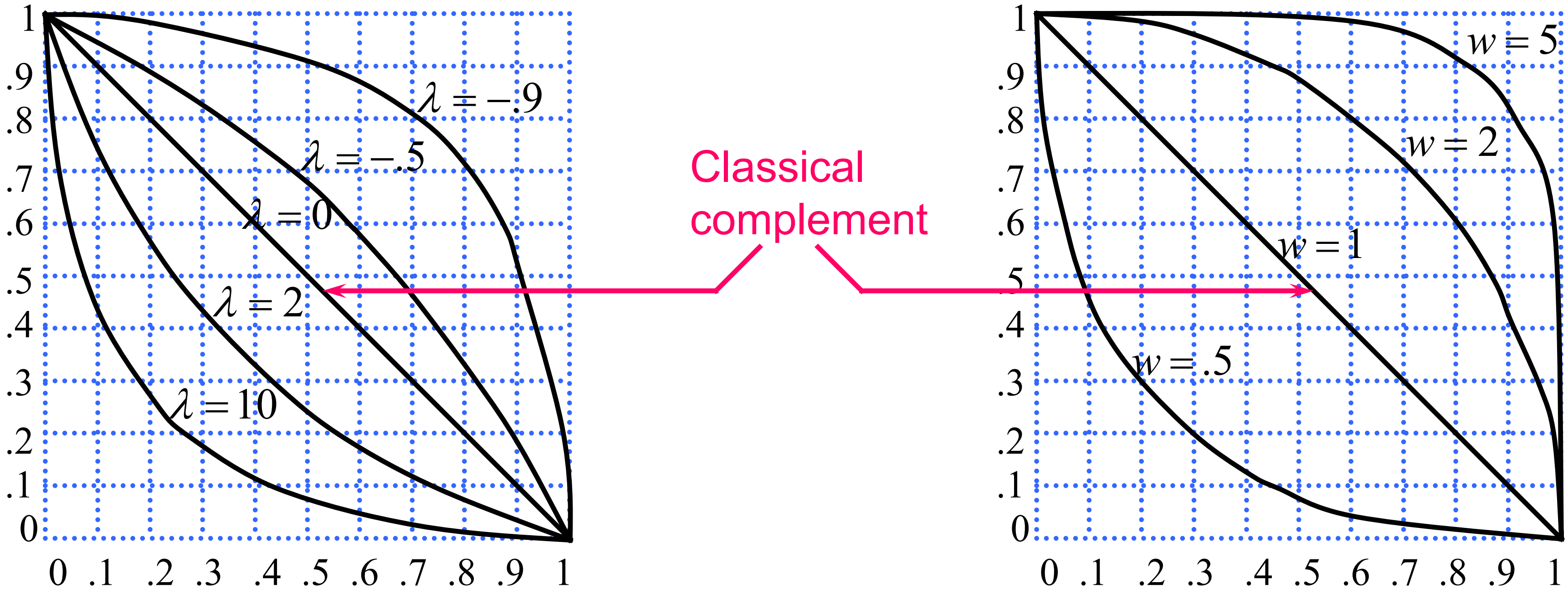




Fuzzy Operations – Complement – e.g.

More examples

- **3.,** $c_\lambda(a)=\dfrac{1-a}{1+\lambda a}$ $\lambda\in(-1,\infty)$ **(M. Sugeno)**
- **2.,** $c_w(a)=\left(1-a^w\right)^{\frac{1}{w}}$ $w\in(0,\infty)$ **(R. Yager)**



Classical complement

Fuzzy Operations – Intersection (t-norm)

$t:[0,1]\times[0,1]\rightarrow[0,1] \qquad \mu_{A\cap B}(x)=t[\mu_A(x),\mu_B(x)]$

- **Axiomatic skeleton:**

t1 $t(a,1)=a \quad \forall a\in[0,1]$ **(boundary conditions)**

t2 $t(a,b)=t(b,a) \quad \forall a,b\in[0,1]$ **(commutativity)**

t3 $b\le c\Rightarrow t(a,b)\le t(a,c)\,\forall a,b,c\in[0,1]$ **(monotonicity)**

t4 $t[t(a,b),c]=t[a,t(b,c)]\,\forall a,b,c\in[0,1]$ **(associativity)**

- **Some usual restrictions (practical motivation)**

t5 ***t* is a continuous function**

t6a $t(a,a)=a$ **(idempotence)**

t6b $t(a,a)<a$ **(subidempotence)**

t7 $a<a'\wedge b<b'\Rightarrow t(a,b)<t(a',b')\,\forall a,b,a',b'\in[0,1]$

(szigorú monotonitás)

Fuzzy Operations – Union (t-conorm, s-norm)

$$s:[0,1]\times[0,1]\to[0,1] \qquad \mu_{A\cup B}(x)=s[\mu_A(x),\mu_B(x)]$$

- **Axiomatic skeleton:**

s1 $s(a,0)=a \quad \forall a\in[0,1]$ **(boundary conditions)**

s2 $s(a,b)=s(b,a) \quad \forall a,b\in[0,1]$ **(commutativity)**

s3 $b\le c\Rightarrow s(a,b)\le s(a,c) \quad \forall a,b,c\in[0,1]$ **(monotonicity)**

s4 $s[s(a,b),c]=s[a,s(b,c)] \quad \forall a,b,c\in[0,1]$ **(associativity)**

- **Some usual restrictions (practical motivation)**

s5 ***s*** **is a continuous function**

s6a $s(a,a)=a$ **(idempotence)**

s6b $s(a,a)>a$ **(superidempotence)**

s7 $a<a'\wedge b<b'\Rightarrow s(a,b)<s(a',b') \quad \forall a,b,a',b'\in[0,1]$

(szigorú monotonitás)

Fuzzy Operations – t-norm, s-norm – e.g.

- **Intersection**

1., $t(a,b)=\min(a,b)$ (Minimum)

2., $t(a,b)=ab$ (Algebraic product)

3., $t(a,b)=\max(0,a+b-1)$ (Bounded product)

4., (Drastic product)

$$t_{\min}(a,b)=\begin{cases} a, & if\ b=1 \\ b, & if\ a=1 \\ 0, & otherwise \end{cases}$$

- **Union**

1., $s(a,b)=\max(a,b)$ (Maximum)

2., $s(a,b)=a+b-ab$ (Algebraic sum)

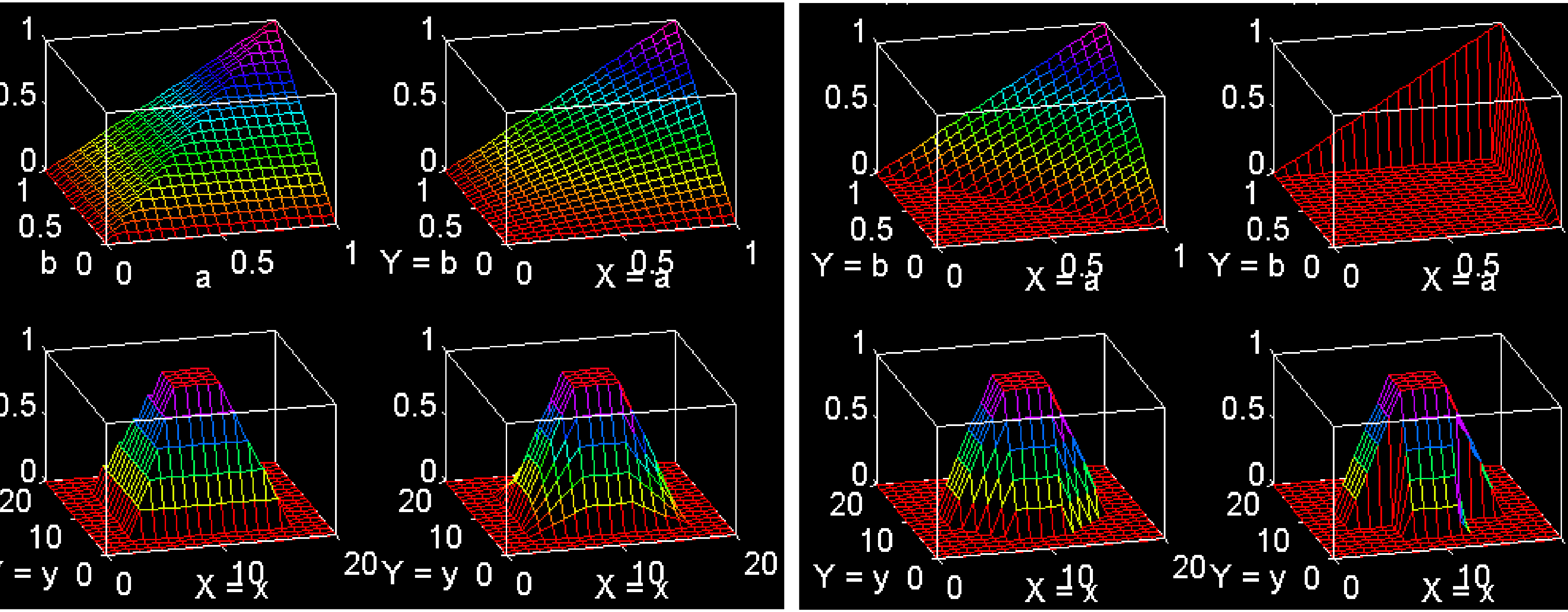
3., $s(a,b)=\min(1,a+b)$ (Bounded sum)

4., (Drastic sum)

$$s_{\max}(a,b)=\begin{cases} a, & if\ b=0 \\ b, & if\ a=0 \\ 1, & otherwise \end{cases}$$

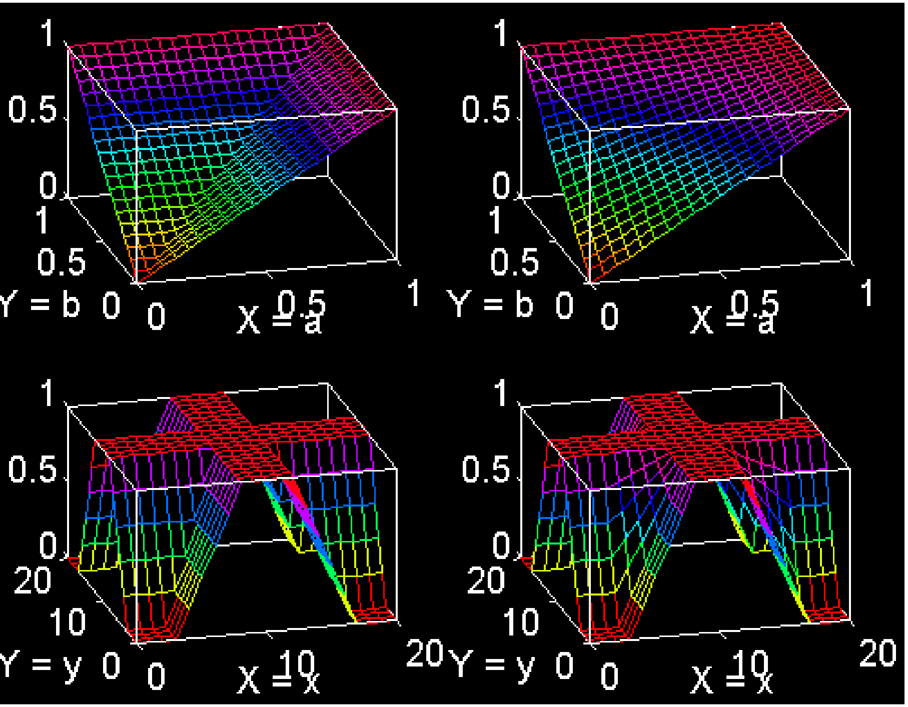
Fuzzy Operations – t-norm – e.g.

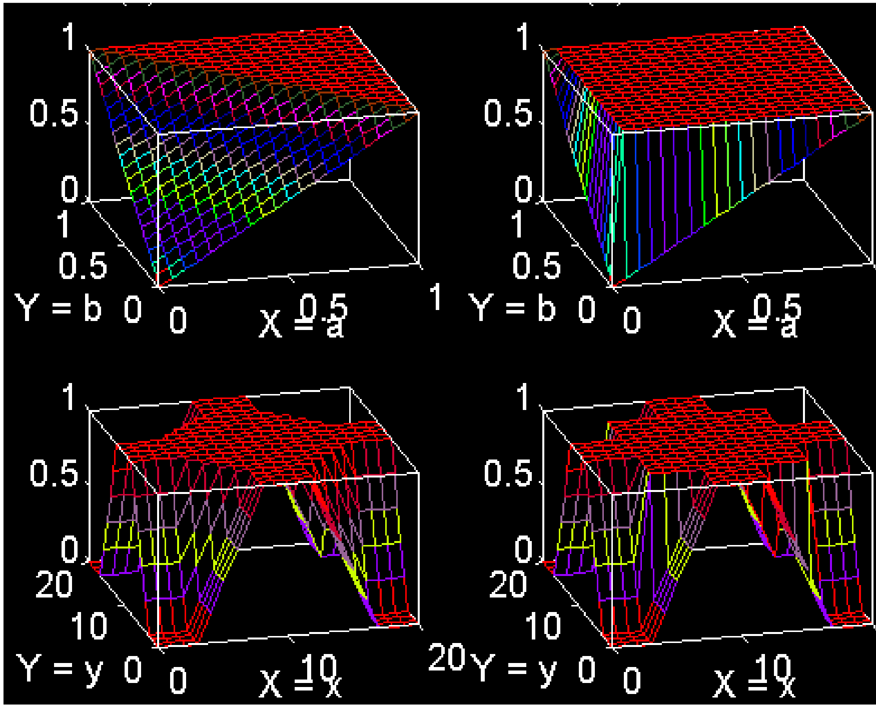
Minimum Algebraic prod. Bounded prod. Drastic prod



Fuzzy Operations – t-conorm, s-norm – e.g.

Maximum Algebraic sum Bounded sum Drastic sum





Fuzzy t-norm, s-norm – some classes

Reference	Fuzzy Unions s-norm	Fuzzy Intersections t-norm	Range of Parameter
Schweizer & Sklar [1961]	$1-\max\left[0,(1-a)^{-p}+(1-b)^{-p}-1\right]^{\frac{1}{p}}$	$\max\left[0,a^{-p}+b^{-p}-1\right]^{-\frac{1}{p}}$	$p\in(-\infty,\infty)$
Hamacher [1978]	$\frac{a+b-(2-\gamma)ab}{1-(1-\gamma)ab}$	$\frac{ab}{\gamma-(1-\gamma)(a+b-ab)}$	$\gamma\in(0,\infty)$
Frank [1979]	$1-\log_s\left[1+\frac{\left(s^{1-a}-1\right)\left(s^{1-b}-1\right)}{s-1}\right]$	$\log_s\left[1+\frac{\left(s^{a}-1\right)\left(s^{b}-1\right)}{s-1}\right]$	$s\in(0,\infty)$
Yager [1980]	$\min\left[1,\left(a^w+b^w\right)^{\frac{1}{w}}\right]$	$1-\min\left[1,\left((1-a)^w+(1-b)^w\right)^{\frac{1}{w}}\right]$	$w\in(0,\infty)$
Dubois & Prade [1980]	$\frac{a+b-ab-\min(a,b,1-\alpha)}{\max(1-a,1-b,\alpha)}$	$\frac{ab}{\max(a,b,\alpha)}$	$\alpha\in(0,1)$
Dombi [1982]	$\frac{1}{1+\left[\left(\frac{1}{a}-1\right)^{-\lambda}+\left(\frac{1}{b}-1\right)^{-\lambda}\right]^{-\frac{1}{\lambda}}}$	$\frac{1}{1+\left[\left(\frac{1}{a}-1\right)^{\lambda}+\left(\frac{1}{b}-1\right)^{\lambda}\right]^{\frac{1}{\lambda}}}$	$\lambda\in(0,\infty)$

Include algebraic norms: $a+b-ab$ **and** ab $p\rightarrow 0$

and Lukasiewicz/Zadeh **max(a,b)** **and min(a,b)** $p\rightarrow -\infty$, $w\rightarrow\infty$

Fuzzy Operations – Aggregation operations

$h:[0,1]^n \to [0,1] \quad n \geq 2 \qquad \mu_A(x) = h(\mu_{A_1}(x), \mu_{A_2}(x), \ldots, \mu_{A_n}(x))$

- **Axiomatic skeleton:**

h1 $h(0,0,\ldots,0) = 0$ **(boundary conditions)**

$h(1,1,\ldots,1) = 1$

h2 for arbitrary *a*$_i$ and *b*$_i$ $i \in N_n$ **(monotonicity)**

$$\forall i \quad a_i \geq b_i \Rightarrow h(a_i | i \in N_n) \geq h(b_i | i \in N_n)$$

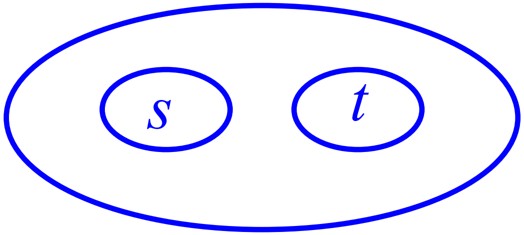
- **Some usual restrictions (practical motivation)**

h3 *h* is a continuous function

h4 *h* is symmetric for all the arguments

$$h(a_i | i \in N_h) = h(a_{p(i)} | i \in N_h)$$

$p(i)$ **arbitrary permutation**

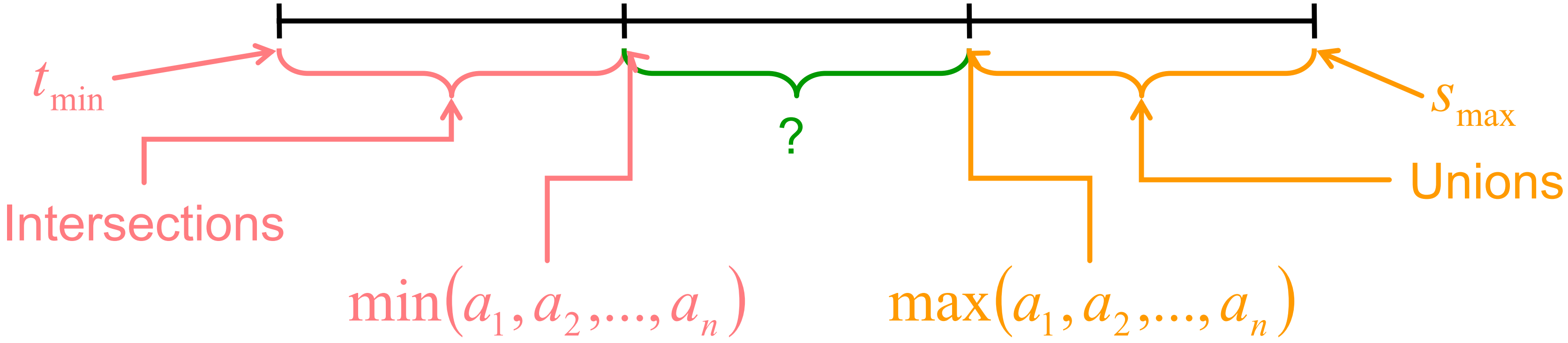


Fuzzy Operations – Aggregation operations

- Union & intersection can be extended to n-ary operations because of associativity:

$$a \cup b \cup c \cup d = (a \cup b) \cup (c \cup d) = ((a \cup b) \cup c) \cup d = \ldots$$

- For given $a_1, a_2, \ldots, a_n$:



- ? is the area of averaging operations

$$\min(a_1, \ldots, a_n) \leq h(a_1, \ldots, a_n) \leq \max(a_1, \ldots, a_n)$$

Fuzzy Operations – Aggregation operations

- **Generalized means:**

$$h_\alpha(a_1,\dots,a_n)=\left(\frac{a_1^\alpha+a_2^\alpha+\dots+a_n^\alpha}{n}\right)^{\frac{1}{\alpha}}$$

h_α **satisfies h1-h4**

$$\alpha\in R\,(\neq 0)$$

$$h_{-\infty}=\min(a_1,a_2,\dots,a_n)$$

$$h_{\infty}=\max(a_1,a_2,\dots,a_n)$$

$$h_0=\sqrt[n]{a_1a_2\dots a_n}$$

$$h_1=\frac{a_1+a_2+\dots+a_n}{n}$$

$$h_{-1}=\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}}$$

Konvex kombináció:

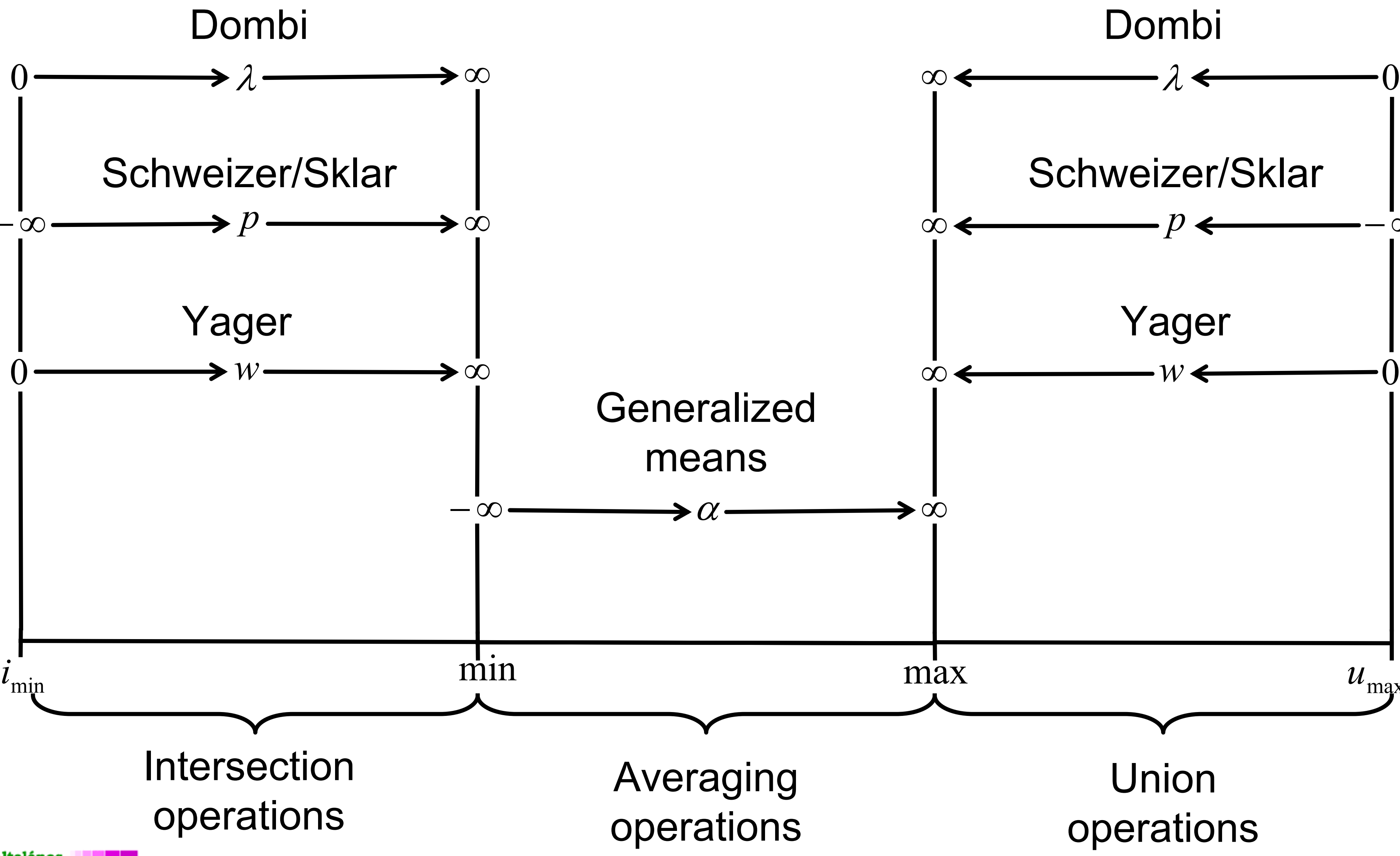
$$h_\alpha^w=\left(\sum_{i=1}^{n}w_ia_i^\alpha\right)^{\frac{1}{\alpha}} \qquad \sum_{i=1}^{n}w_i=1$$

If h4 (symmetricity) **is not necessar**

(different importance of arguments)

Fuzzy Operations – Aggregation operations

- **Various classes of aggregation operations**



Ajánlott irodalom

- **The slides of this lecture are partially based on the books:**

Kóczy T. László és Tikk Domonkos: ***Fuzzy rendszerek*****,** Typotex Kiadó, 2000, **ISBN 963-9132-55-1**

J.-S. R. Jang, C.-T. Sun, E. Mizutani: ***Neuro-Fuzzy and Soft Computing*****,** Prentice Hall, 1997, **ISBN 0-13-261066-3**

Michael Negnevitsky: ***Artificial Intelligence: A Guide to Intelligent Systems*****,** Addison Wesley, Pearson Education Limited, 2002, **ISBN 0201-71159-1**