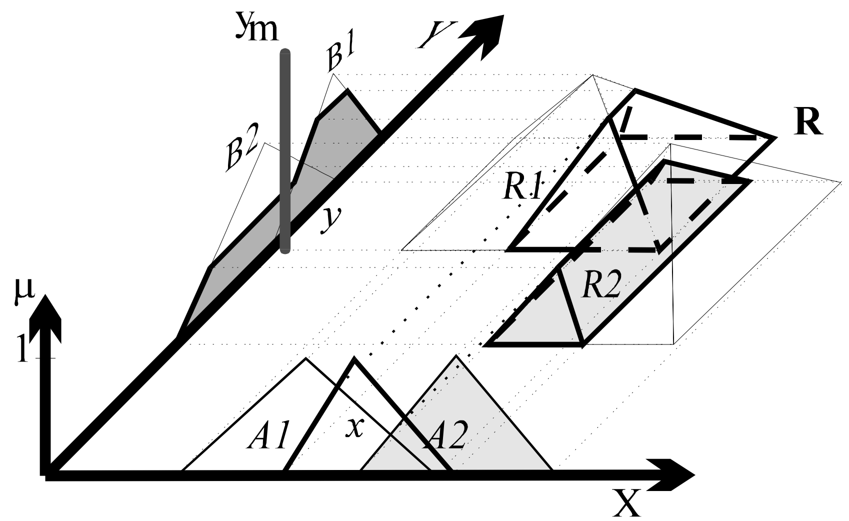
A fuzzy rulebase is **complete**, if the union of the antecedent parts of the fuzzy rules are covering all the input universe:

$$\bigcup_{i=1}^{r} \mathrm{supp}(A_i) = \boldsymbol{X}$$

where $\boldsymbol{R} = \{R_1, \ldots, R_r\}$ the fuzzy rulebase,

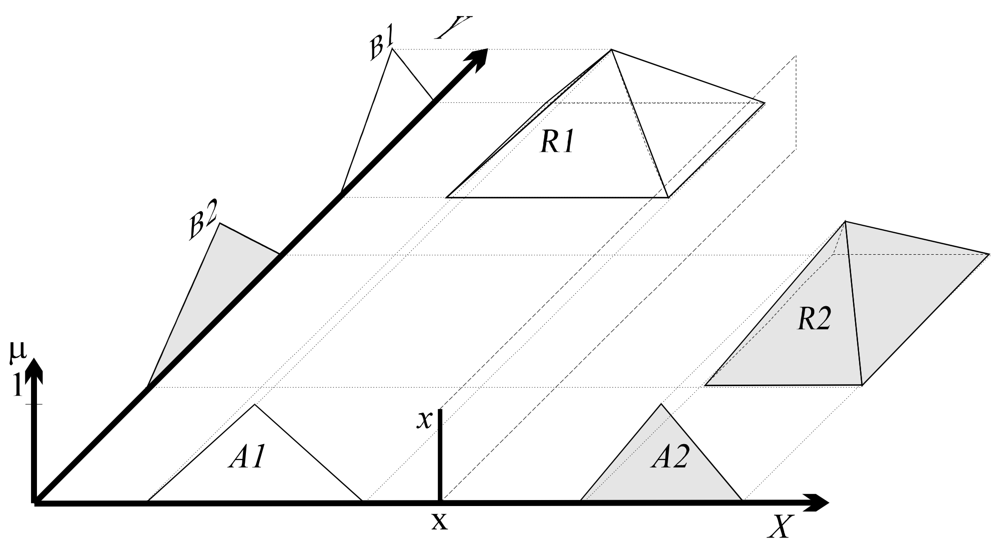
$R_i = A_i \to B_i$ the i. fuzzy rule,

$\boldsymbol{X} = X_1 \times X_2 \times \ldots \times X_n$ the input universe of discourse.



If the rulebase is **sparse** (not complete), there are an observation $\boldsymbol{x}$ exists, which does not hit any of the rules:

$$\bigcup_{i=1}^{r} \mathrm{supp}(A_i) \subset X \quad \Rightarrow \quad x \cap \bigcup_{i=1}^{r} \mathrm{supp}(A_i) = \varnothing$$



The concept of **vague environment** is based on the **similarity** or **indistinguishability** of the elements.

Two values in the vague environment X are **ε-indistinguishable** if

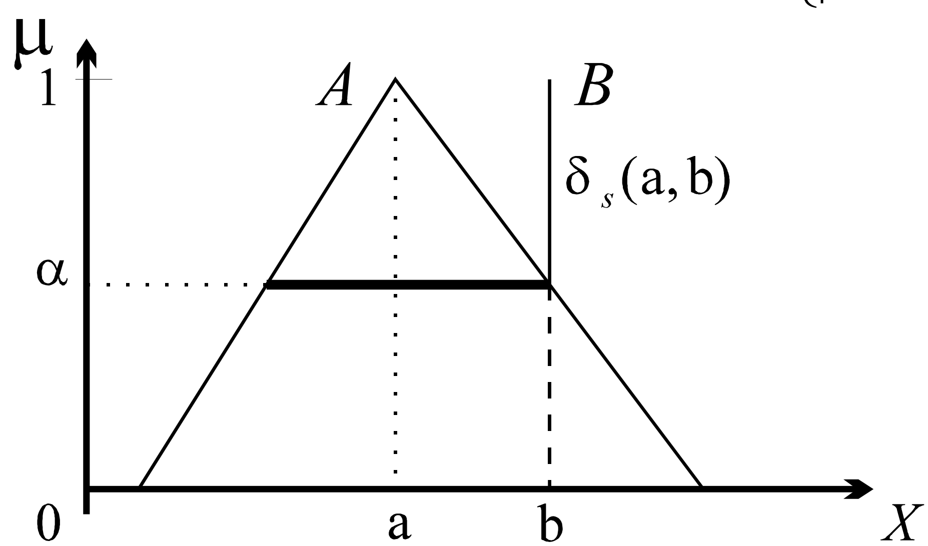
$$\varepsilon \geq \delta_s(x_1, x_2) = \left| \int_{x_2}^{x_1} s(x)\,dx \right|$$

where $\delta_s(x_1, x_2)$ is the **vague distance** of the values x_1, x_2,

$s(x)$ is the **scaling function** on X [Klawonn].

We can introduce the **membership function** $\mu_A(x)$ as a level of similarity, as the degree to which x is indistinguishable to **a** [Klawonn]. The **α-cuts** of the fuzzy set $\mu_A(x)$ is the set which contains the elements that are **(1−α)-indistinguishable** from **a**:

$$\delta_s(\mathrm{a}, \mathrm{b}) \leq 1 - \alpha, \qquad \mu_A(x) = 1 - \min\left\{ \left| \int_{\mathrm{a}}^{x} s(x)\,dx \right|, 1 \right\}$$



The vague distance of points - **Disconsistency Measure** (S_D) of the fuzzy sets A and B (where B is a singleton):

$$S_D = 1 - \sup_{x \in X} \mu_{A \cap B}(x) = \delta_s(\mathrm{a}, \mathrm{b}) \quad \text{if} \quad \delta_s(\mathrm{a}, \mathrm{b}) \in [0, 1]$$

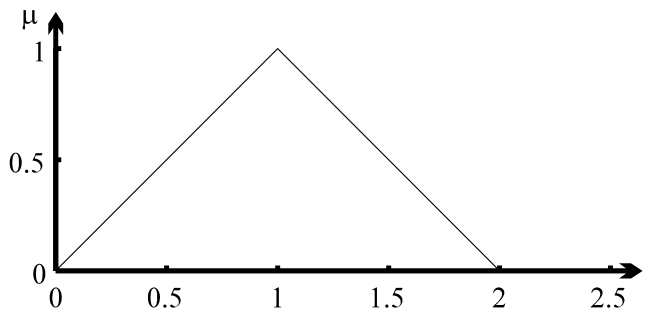
where $A \cap B$ is the min t-norm, $\mu_{A \cap B}(x) = \min[\mu_A(x), \mu_B(x)] \;\; \forall\, x \in X$.

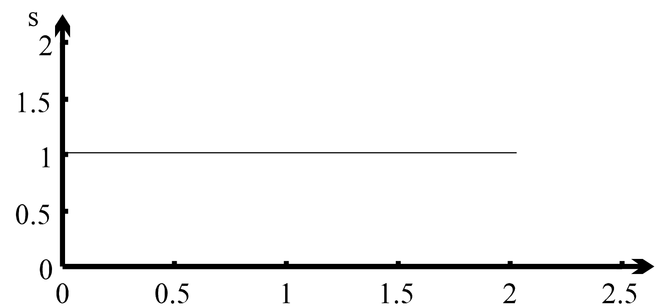
Vague environments - fuzzy partitions

The vague environment is described by its scaling function.

For generating a vague environment we have to find an appropriate scaling function, which describes the shapes of all the terms in the fuzzy partition.

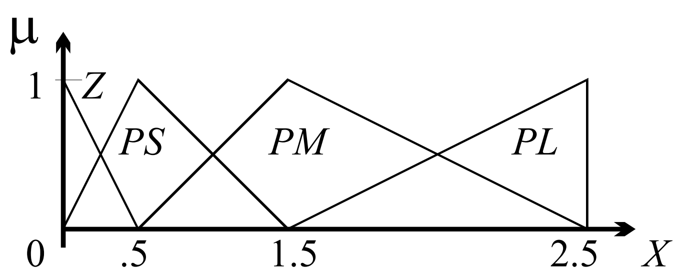
$$s(x) = \left|\mu'(x)\right| = \left|\frac{d\mu}{dx}\right| \quad \text{[Klawonn]}$$

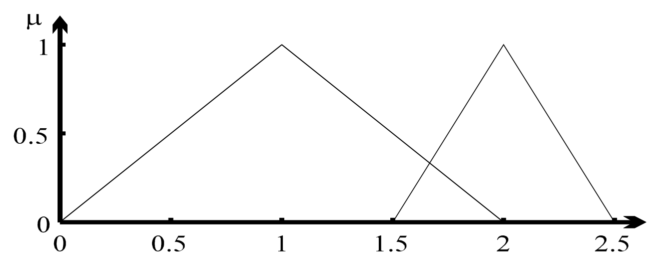


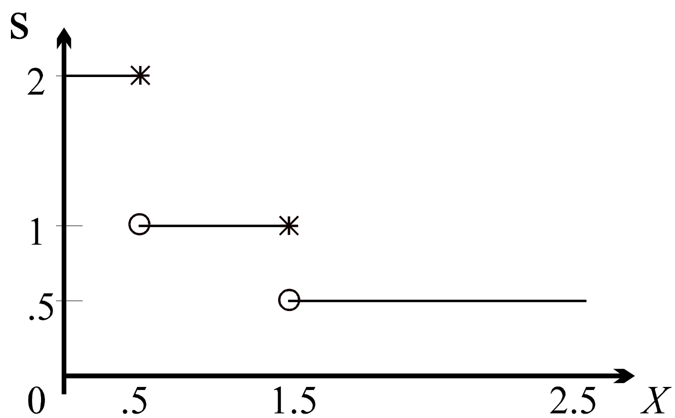


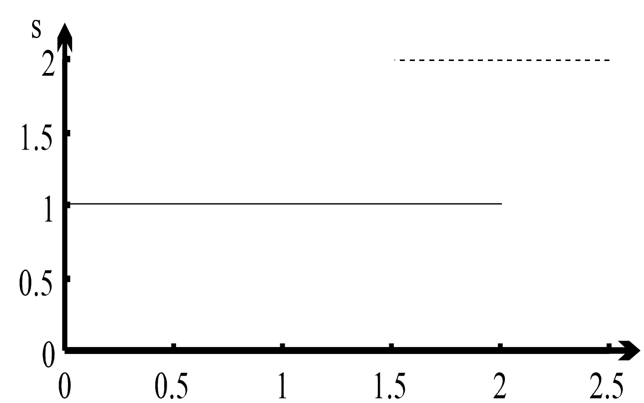
Restriction for the membership functions of the terms [Klawonn] :

$$\min\left\{\mu_i(x), \mu_j(x)\right\} > 0 \;\Rightarrow\; \left|\mu'_i(x)\right| = \left|\mu'_j(x)\right| \qquad \forall i, j \in I$$









There are a **scaling function exists**, which describes all the fuzzy sets

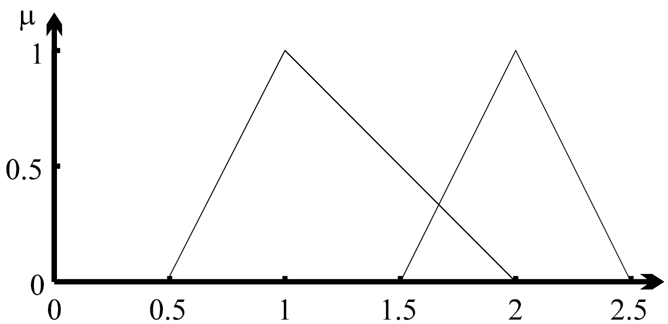
There are **no** scaling function exists, which describes both the fuzzy sets

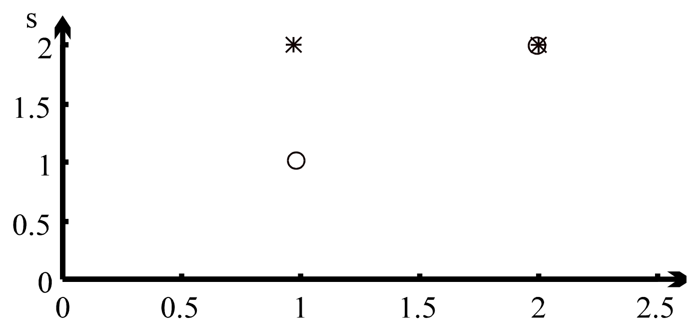
The question is how to describe all fuzzy sets of the fuzzy partition with one "universal" scaling function.

The approximate scaling function

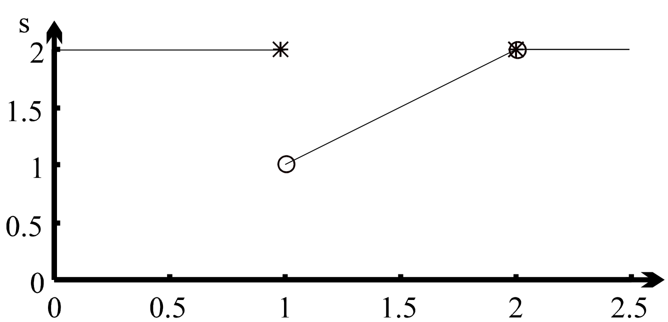
The approximate scaling function is an approximation of the scaling functions describes the terms of the fuzzy partition separately.

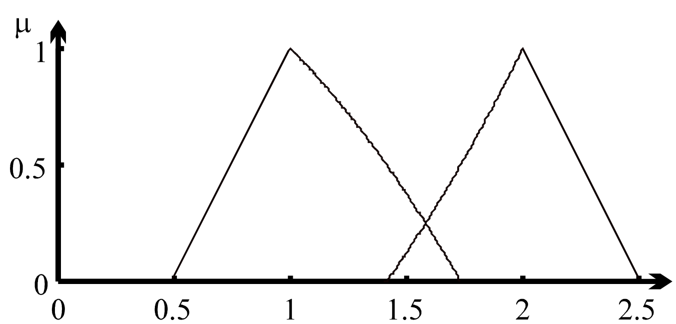
Supposing that the fuzzy terms are **triangles**, each fuzzy term can be characterised by two constant, the scaling factor of the left and the right slope of the triangle. So a triangle shaped fuzzy term can be characterised by **three values** (by a triple), by the values of the





The simplest way of generating the approximate scaling function is the **linear interpolation of the scaling function** between the neighbouring terms.





The main problem of the linearly interpolated scaling functions, that they cant handle the big differences in neighbouring scaling factors or crisp fuzzy sets correctly. If there are big differences in the neighbouring scaling factors, the bigger scaling factor is "dominating" the smaller factor.

The proposed **non-linear interpolation** for the approximate scaling function:

$$s(x)=\begin{cases}\dfrac{w_i}{(d_i+1)^{k\cdot w_i}-1}\cdot\left(\dfrac{(d_i+1)^{k\cdot w_i}}{(x-x_i+1)^{k\cdot w_i}}-1\right)+s_{i+1}^L & \middle|\ s_i^R\ge s_{i+1}^L,\\[2ex] \dfrac{w_i}{(d_i+1)^{k\cdot w_i}-1}\cdot\left(\dfrac{(d_i+1)^{k\cdot w_i}}{(x_{i+1}-x+1)^{k\cdot w_i}}-1\right)+s_i^R & \middle|\ s_i^R< s_{i+1}^L,\end{cases}$$

$$x\in[x_i,x_{i+1}),\ \forall\, i\in[1,n-1]$$

where

$$w_i=\left|s_{i+1}^L-s_i^R\right|,\ \forall\, i\in[1,n-1],\quad d_i=x_{i+1}-x_i,\ \forall\, i\in[1,n-1]$$

$s(x)$ is the approximate scaling function
x_i is the core of the i^{th} term
s_i^L, s_i^R are the i^{th} left and right side scaling factors
k constant factor of sensitivity of the differences
n is the number of the terms in the fuzzy partition

Useful properties:

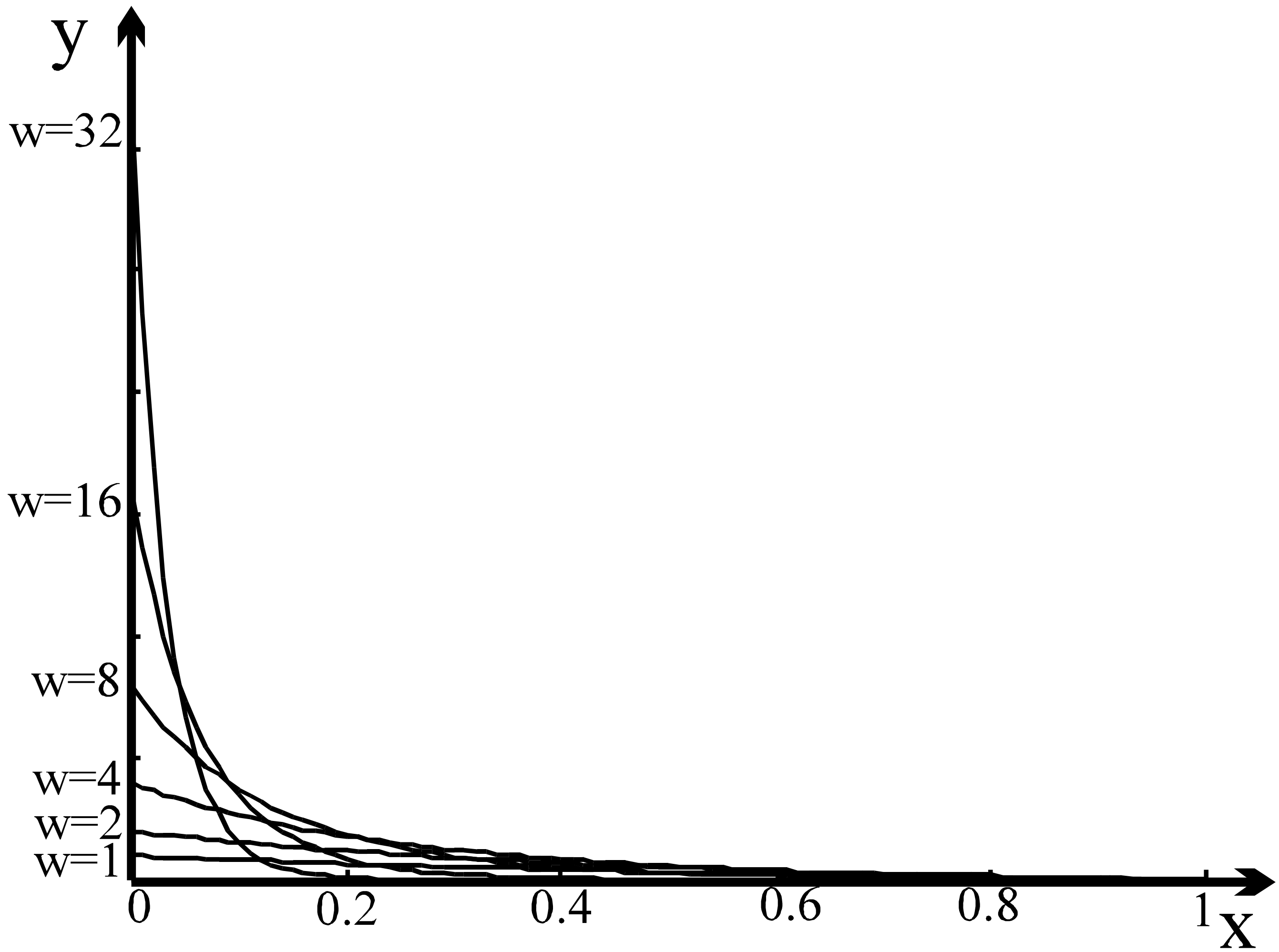
If the neighbouring scaling factors are equals, $s(x)$is linear

$$s_i^R=s_{i+1}^L\ ,\ x\in[x_i,x_{i+1})\ \Rightarrow\ s(x)=s_i^R=s_{i+1}^L$$

If one of the neighbouring scaling factors is infinite e.g. $s_i^R\to\infty$ (the right side of the i^{th} term is crisp) and s_{i+1}^L finite then

$$s(x)\to\begin{cases}\infty \mid x=x_i\\ s_{i+1}^L\ \text{otherwise}\end{cases}\qquad x\in[x_i,x_{i+1})$$

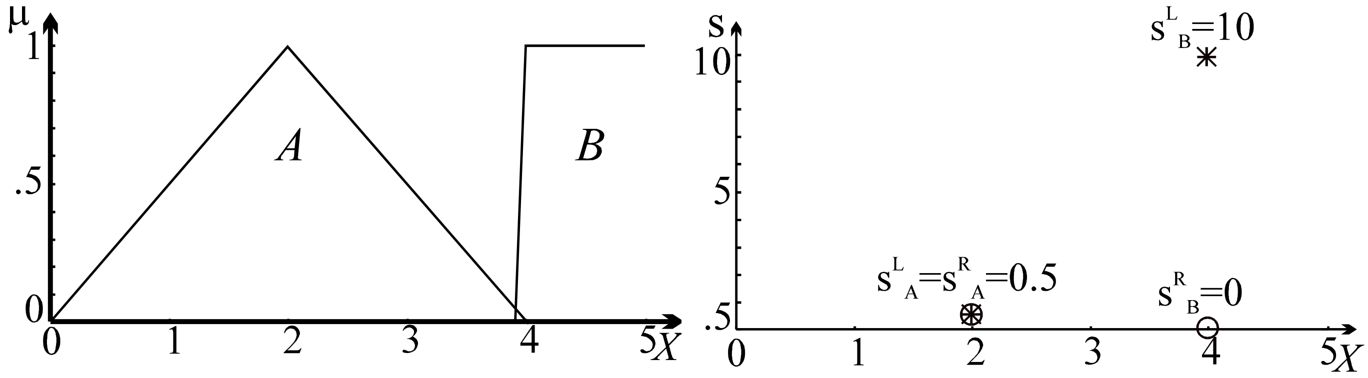
Examples for the proposed non-linear function



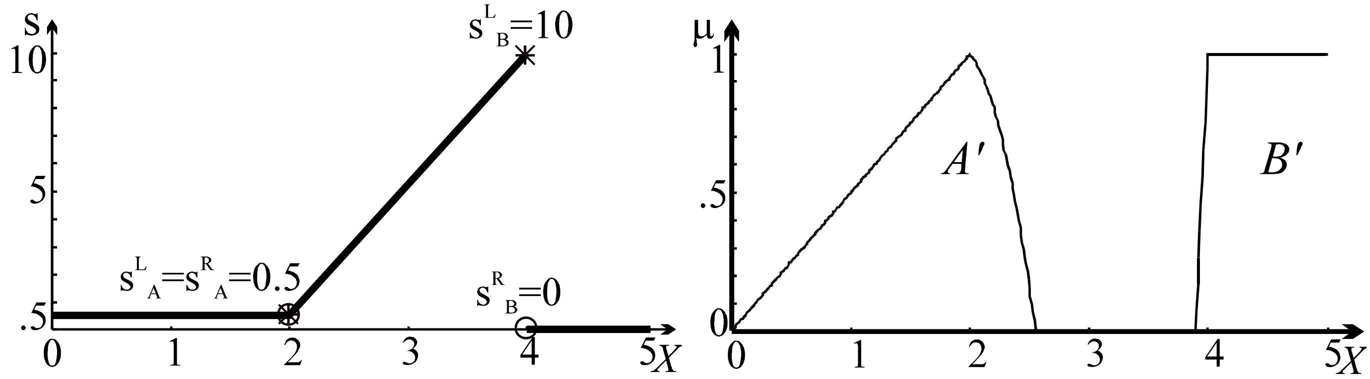
$x_1=0$, $x_2=1$, $k=1$,

$s_1^R \geq s_2^L$, $s_1^R = w$, $s_2^L = 0$,

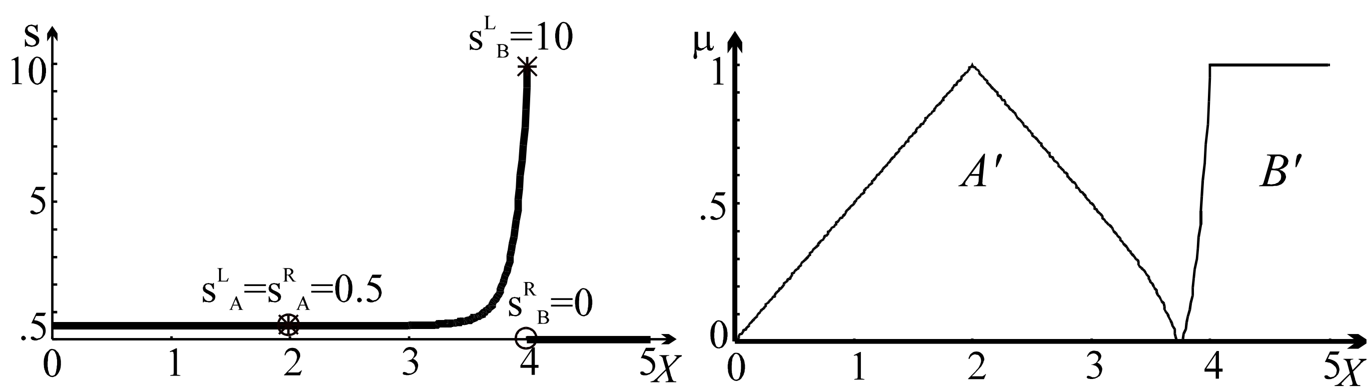
$$s(x) = \frac{w}{2^{k \cdot w} - 1} \cdot \left(\frac{2^{k \cdot w}}{(x+1)^{k \cdot w}} - 1 \right) \approx \frac{1}{x^{k \cdot w}}$$



Big differences in neighbouring scaling factor values ($s_A^R \ll s_B^L$)



Linearly interpolated scaling function, the bigger scaling factor is "dominating" the smaller factor (*A'*,*B'*)



Approximate scaling function generated by the proposed non-linear function (*k*=1), describing the fuzzy partition (*A'*,*B'*)