APPROXIMATE FUZZY REASONING BASED ON INTERPOLATION IN THE VAGUE ENVIRONMENT OF THE FUZZY RULEBASE AS A PRACTICAL ALTERNATIVE OF THE CLASSICAL CRI

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Abstract

Using the concept of vague environment described by scaling functions [2] instead of the linguistic terms of the fuzzy partition gives a simple way for fuzzy approximate reasoning. In most of the practical applications, the fuzzy partitions (used as primary sets of the fuzzy rulebase) can be described by vague environments (based on the similarity or indistinguishability of the elements [2]). Comparing a description of a universe given by a fuzzy partition to the way of using the concept of vague environment we can say, that the linguistic terms of the fuzzy partition are points in the vague environment, while the shape of the fuzzy sets is described by the scaling function. This case the primary fuzzy sets of the antecedent and the consequent parts of the fuzzy rules are points in their vague environments, so the fuzzy rules themselves are points in their vague environment too (in the vague environment of the fuzzy rulebase). It means, that the question of approximate fuzzy reasoning can be reduced to the problem of interpolation of the rule points in the vague environment of the fuzzy rulebase relation [4,5]. In other words, using the concept of vague environment, in most of the practical cases we can build approximate fuzzy reasoning methods simple enough to be a good alternative of the classical Compositional Rule of Inference methods in practical applications. In this paper two methods of approximate fuzzy reasoning based on interpolation in the vague environment of the fuzzy rulebase, and a comparison of these methods to the classical CRI will be introduced.

1. Connection between similarity of fuzzy sets and vague distance of points in a vague environment

The concept of vague environment is based on the similarity or indistinguishability of the elements. Two values in the vague environment are ε-distinguishable if their distance is grater then ε. The distances in the vague environment are weighted distances. The weighting factor or function is called scaling function (factor) [2]. Two values in the vague environment X are ε-distinguishable if

$$\varepsilon > \delta_s(x_1, x_2) = \left| \int_{x_2}^{x_1} s(x)dx \right|$$

where $\delta_s(x_1, x_2)$ is the vague distance of the values x_1, x_2 and $s(x)$ is the scaling function on X.

For finding connections between fuzzy sets and a vague environment we can introduce the membership function $\mu_A(x)$ as a level of similarity **a** to x, as the degree to which x is indistinguishable to **a** [2]. The α-cuts of the fuzzy set $\mu_A(x)$ is the set which contains the elements that are $(1-\alpha)$-indistinguishable from **a** (see fig.1.):

$$\delta_s(a, b) \le 1 - \alpha$$

$$\mu_A(x) = 1 - \min\{\delta_s(a, b), 1\} = 1 - \min\left\{\left|\int_a^b s(x)dx\right|, 1\right\}$$

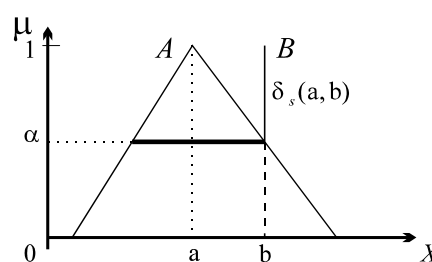
Fig.1. The α-cuts of $\mu_A(x)$ contains the elements that are $(1-\alpha)$-indistinguishable from **a**

It is very easy to realise (see fig.1.), that this case the vague distance of points a and b ($\delta_s(a, b)$) is basically the *Disconsistency Measure* (S_D) of the fuzzy sets A and B (where B is a singleton):

$$S_D = 1 - \sup_{x \in X} \mu_{A \cap B}(x) = \delta_s(a, b)$$

$$\text{if } \delta_s(a, b) \in [0, 1]$$

where $A \cap B$ is the min t-norm, $\mu_{A \cap B}(x) = \min\left[\mu_A(x), \mu_B(x)\right] \ \forall\, x \in X$.

It means, that we can calculate the disconsistency measures between member fuzzy sets of a fuzzy partition and a singleton, as vague distances of points in the vague environment of the fuzzy partition. The main difference between the disconsistency measure and the vague distance is, that the vague distance is a crisp value in range of $[0,\infty]$, while the disconsistency measure is limited to $[0,1]$. That is why they are useful in interpolate reasoning with insufficient evidence.

So if it is possible to describe all the fuzzy partitions of the primary fuzzy sets (the antecedent and consequent universes) of our fuzzy rulebase, and the observation is a singleton, we can calculate the "extended" disconsistency measures of the antecedent primary fuzzy sets of the rulebase and the observation, and the "extended" disconsistency measures of the consequent primary fuzzy sets and the consequence (we are looking for) as vague distances of points in the antecedent and consequent vague universes.

2. **Generating vague environments from the fuzzy partitions of the linguistic terms of the fuzzy rules**

The vague environment is described by its scaling function. For generating a vague environment we have to find an appropriate scaling function, which describes the shapes of all the terms in the fuzzy partition.
The method proposed by Klawonn [2], for choosing the scaling function $s(x)$, gives an exact description of the fuzzy terms after their reconstruction from the scaling function:

$$s(x) = \left|\mu'(x)\right| = \left|\frac{d\mu}{dx}\right|$$

So we always find a scaling function, if we have only one fuzzy set in the fuzzy partition. Usually the fuzzy partition contains more than one fuzzy set, so this method requires some restrictions for the membership functions of the terms [2]:

$$s(x) = \left|\mu'(x)\right| = \left|\frac{d\mu}{dx}\right| \quad \text{exists} \qquad \text{iff}$$

$$\min\left\{\mu_i(x), \mu_j(x)\right\} > 0 \ \Rightarrow \ \left|\mu'_i(x)\right| = \left|\mu'_j(x)\right|$$

$$\forall i,j \in I$$

Generally the above condition is not fulfilling, so the question is how to describe all

fuzzy sets of the fuzzy partition with one "universal" scaling function. For this reason we propose to use the **approximate scaling function**.

3. **The approximate scaling function**

The approximate scaling function is an approximation of the scaling functions describes the terms of the fuzzy partition separately.
Supposing that the fuzzy terms are **triangles**, each fuzzy term can be characterised by two constant scaling functions, the scaling factor of the left and the right slope of the triangle. So a triangle shaped fuzzy term can be characterised by three values (by a triple), by the values of the left and the right scaling factors and the value of its core point (e.g.fig.2.).
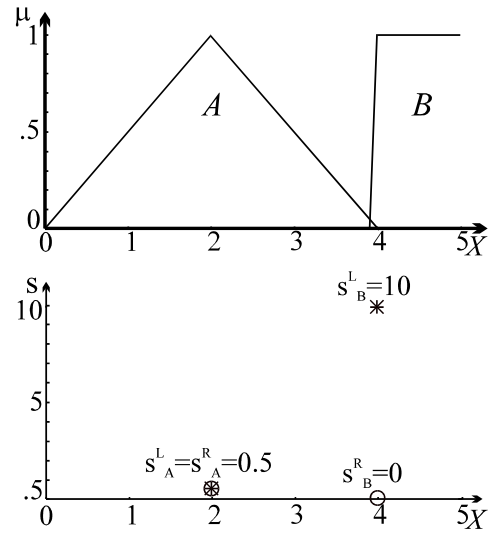

Fig.2. Two triangle shaped fuzzy set characterised by two triple, by the left and the right scaling factors and the value of the core point

For generating the approximate scaling function we suggest to adopt the following non-linear function for interpolating the neighbouring scaling factors:

$$s(x) = \begin{cases} \dfrac{w_i}{\left(d_i+1\right)^{k \cdot w_i} - 1} \cdot \left(\dfrac{\left(d_i+1\right)^{k \cdot w_i}}{\left(x - x_i + 1\right)^{k \cdot w_i}} - 1\right) + s_{i+1}^{L} \\[4pt] \quad \left| \ s_i^{R} \geq s_{i+1}^{L}, \right. \\[10pt] \dfrac{w_i}{\left(d_i+1\right)^{k \cdot w_i} - 1} \cdot \left(\dfrac{\left(d_i+1\right)^{k \cdot w_i}}{\left(x_{i+1} - x + 1\right)^{k \cdot w_i}} - 1\right) + s_i^{R} \\[4pt] \quad \left| \ s_i^{R} < s_{i+1}^{L}, \right. \end{cases}$$

$$x \in [x_i, x_{i+1}), \ \forall\, i \in [1, n-1]$$

where

$$w_i = \left| s_{i+1}^{L} - s_i^{R} \right|, \ \forall\, i \in [1, n-1],$$

$d_i = x_{i+1} - x_i, \ \forall \ i \in [1, n-1]$,

$s(x)$ is the approximate scaling function, x_i is the core of the ith term of the approximated fuzzy partition, s_i^L, s_i^R are the left and right side scaling factors of the ith triangle shaped term of the approximated fuzzy partition, k constant factor of sensitivity for neighbouring scaling factor differences, n is the number of the terms in the approximated fuzzy partition

The above function has the following useful properties:

If the neighbouring scaling factors are equals, $s(x)$ is linear

$$s_i^R = s_{i+1}^L, \ x \in [x_i, x_{i+1}] \ \Rightarrow \ s(x) = s_i^R = s_{i+1}^L$$

If one of the neighbouring scaling factors is infinite e.g. $s_i^R \to \infty$ (the right side of the ith term is crisp) then

$s_i^R \to \infty$ and s_{i+1}^L finite, $x \in [x_i, x_{i+1}) \Rightarrow$

$$s(x) \to \begin{cases} \infty \mid x = x_i \\ 0 \text{ otherwise} \end{cases}$$

similarly

$s_{i+1}^L \to \infty$ and s_i^R finite, $x \in [x_i, x_{i+1}) \Rightarrow$

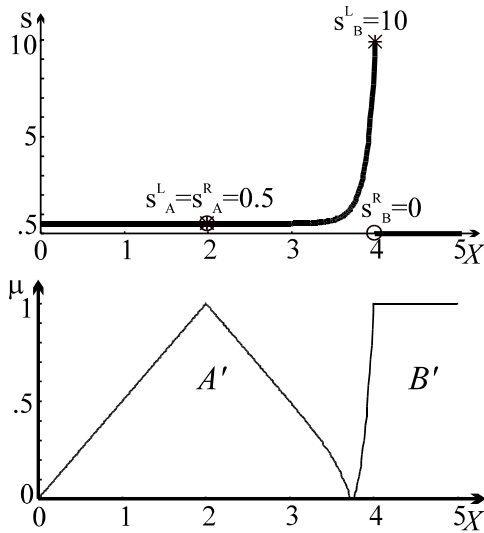$$s(x) \to \begin{cases} \infty \mid x \to x_{i+1} \\ 0 \text{ otherwise} \end{cases}$$

Fig.3. Approximate scaling function generated by the proposed non-linear function (k=1), and the original fuzzy partition (A,B) as this scaling function describes it (A',B')

4. **Calculating the conclusion by approximating the vague points of the rulebase**

If the vague environment of a fuzzy partition (the scaling function or the

approximate scaling function) exists, the member sets of the fuzzy partition can be characterised by points in the vague environment. (In our case the points are characterising the cores of the terms, while the shapes of the membership functions are described by the scaling function.) If all the vague environments of the antecedent and consequent universes of the fuzzy rulebase are exist, all the primary fuzzy sets (linguistic terms) used in the fuzzy rulebase can be characterised by points in their vague environment. So the fuzzy rules (build on the primary fuzzy sets) can be characterised by points in the vague environment of the fuzzy rulebase too. This case the approximate fuzzy reasoning can be handled as a classical interpolation task. Applying the concept of vague environment (the distances of points are weighted distances), any interpolation, extrapolation or regression methods can be adapted very simply for approximate fuzzy reasoning.

For example we can use the **Lagrange interpolation**. The original formula is the following:

$$Y(x) = \frac{(x-x_2) \cdot (x-x_3) \cdot \ldots \cdot (x-x_n)}{(x_1-x_2) \cdot (x_1-x_3) \cdot \ldots \cdot (x_1-x_n)} \cdot y_1 +$$

$$+ \frac{(x-x_1) \cdot (x-x_3) \cdot \ldots \cdot (x-x_n)}{(x_2-x_1) \cdot (x_2-x_3) \cdot \ldots \cdot (x_2-x_n)} \cdot y_2 + \ldots$$

$$\ldots + \frac{(x-x_1) \cdot (x-x_2) \cdot \ldots \cdot (x-x_{n-1})}{(x_n-x_1) \cdot (x_n-x_2) \cdot \ldots \cdot (x_n-x_{n-1})} \cdot y_n$$

where $Y(x)$ is the Lagrange interpolation of the n points $(x_1,y_1), (x_2,y_2),\ldots, (x_n,y_n)$.

Using the concept of vague distances in the case of one dimensional antecedent universe:

$$\text{dist}(y_0, y) =$$

$$= \frac{\text{dist}(a_2, x) \cdot \text{dist}(a_3, x) \cdot \ldots \cdot \text{dist}(a_r, x)}{\text{dist}(a_2, a_1) \cdot \text{dist}(a_3, a_1) \cdot \ldots \cdot \text{dist}(a_r, a_1)} \cdot \text{dist}(y_0, b_1) +$$

$$+ \frac{\text{dist}(a_1, x) \cdot \text{dist}(a_3, x) \cdot \ldots \cdot \text{dist}(a_r, x)}{\text{dist}(a_1, a_2) \cdot \text{dist}(a_3, a_2) \cdot \ldots \cdot \text{dist}(a_r, a_2)} \cdot \text{dist}(y_0, b_2) + \ldots$$

$$\ldots + \frac{\text{dist}(a_1, x) \cdot \text{dist}(a_2, x) \cdot \ldots \cdot \text{dist}(a_{r-1}, x)}{\text{dist}(a_1, a_r) \cdot \text{dist}(a_2, a_r) \cdot \ldots \cdot \text{dist}(a_{r-1}, a_r)} \cdot \text{dist}(y_0, b_r)$$

where

$$\text{dist}(a_k, a_l) = \int_{a_k}^{a_l} s_X(x) dx_i$$

y_0 is the first element of the universe Y:

$y_0 \leq y \quad \forall \ y \in Y$

(as Y is a one dimensional universe)

Another example as an adaptation of a simple **rational interpolation** is the following:

$$\text{dist}\left(y_0, y\right) = \frac{\sum\limits_{k=1}^{r} w_k \cdot \text{dist}\left(y_0, b_k\right)}{\sum\limits_{k=1}^{r} w_k}$$

where

$$w_k = \frac{1}{\left(\text{dist}\left(\mathbf{x}, \mathbf{a}_k\right)\right)^p}$$

is a weighting factor inversely proportional to the vague distance of the observation and the kth rule antecedent,

$$\text{dist}\left(\mathbf{a}_k, \mathbf{x}\right) = \text{dist}\left(\mathbf{x}, \mathbf{a}_k\right) = \sqrt{\sum\nolimits_{i=1}^{m}\left(\int\limits_{a_{k,i}}^{x_i} s_{X_i}\left(x_i\right) dx_i\right)^2} \ ,$$

$$\text{dist}\left(y_0, b_k\right) = \int\limits_{y_0}^{b_k} s_Y\left(y\right) dy \ ,$$

s_{X_i} is the ith scaling function of the m dimensional antecedent universe,

s_Y is the scaling function of the one dimensional consequent universe,

$\mathbf{x}$ is the multidimensional crisp observation,

$\mathbf{a}_k$ are the cores of the multidimensional fuzzy rule antecedents A_k,

b_k are the cores of the one dimensional fuzzy rule consequents B_k,

$R_i = A_i \rightarrow B_i$ are the fuzzy rules,

p is the sensitivity of the weighting factor for distant rules,

y_0 is the first element of the one dimensional universe (Y: $y_0 \leq y \ \forall \ y \in Y$),

y is the one dimensional conclusion we are looking for.

Comparing the two proposed interpolation methods we can establish the following:

Having only two rules, between the two rule antecedents the two methods give the same conclusion (e.g.Fig.4.).

Because of the absolute antecedent distances of the proposed rational interpolation function, the approximate reasoning method based on this function can be used in case of multidimensional antecedent universes too
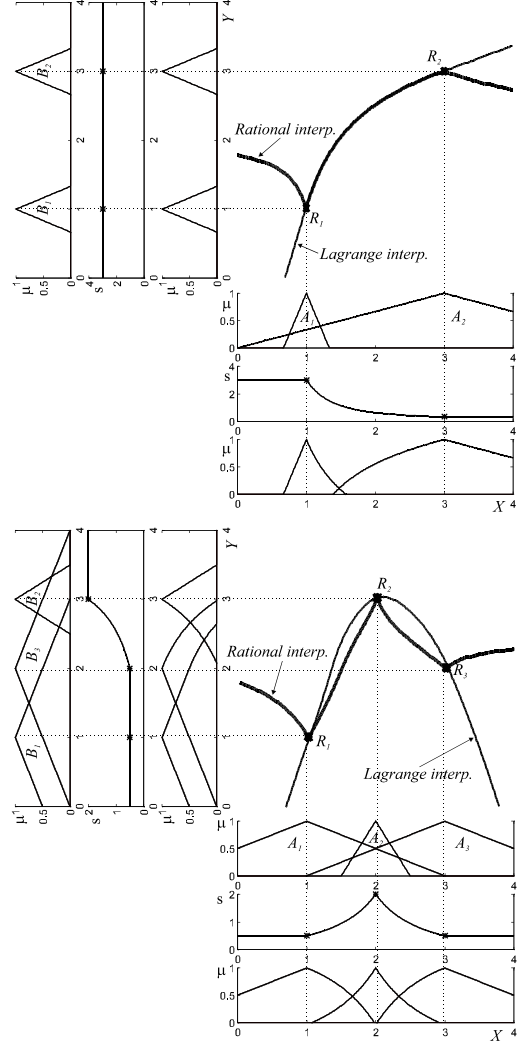

Fig.4. Interpolation of fuzzy rules ($R_i:A_i \rightarrow B_i$) in the approximated vague environment of the fuzzy rulebase, using the proposed **rational interpolation** (p=1) and the adopted **Lagrange interpolation**

5. **Comparing the crisp conclusion generated by approximate reasoning in the vague environment of the fuzzy rulebase to the crisp conclusion generated by the classical Compositional Rule of Inference**

For comparing the crisp conclusions generated by the proposed approximate reasoning method to the classical Compositional Rule of Inference (CRI), we are choosing as a representative one, the min-max. compositional rule of inference and the centre of gravity defuzzification method. Comparing the crisp conclusions of the proposed approximate fuzzy reasoning and the classical CRI (fig.5.), the most striking difference is, that the control function of the approximate fuzzy reasoning is always fits the points of the fuzzy rules. (This is a property of the

interpolation function used for the approximate reasoning.) While, the control function of the CRI is usually not fits these points. In practical sense it means that, if an observation hits a rule antecedent exactly, than the conclusion generated by approximate fuzzy reasoning will be equal to the consequent part of the same fuzzy rule. The next difference is the main reason of the approximate fuzzy reasoning methods for insufficient evidence. The approximate fuzzy reasoning method gives conclusion for all the observations of the antecedent universe, even if the fuzzy rulebase is not complete, while the CRI gives no conclusion if there are no overlapping between the observation and at least one of the rule antecedents. (Fig.5.) The last mentioned difference is a kind of philosophical question. The wide rule consequents has more influence to the defuzzified crisp conclusion of the CRI, because of the "wide consequents" are more "heavy" in the fuzzy conclusion (using the centre of gravity defuzzification). While using the method based on approximation in the vague environment of the fuzzy rulebase, the situation is the opposite. (Fig.5.) The idea we used, that the rate of the distances of the observation and the rule antecedents must be equal to the rate of the distances between the conclusion and the corresponding rule consequents, has a special property in case of using the concept of vague environment for calculating the distances of fuzzy sets. If a rule consequent is "narrower than the other", the scaling function is higher there, the surrounding vague environment is more "dense". It means smaller distances in the consequent universe. So the narrow rule consequent is dominating the "wider" ones. In other words it means, that using the CRI, in the crisp conclusion those rules has dominance, whose consequent part is more "global" (more "imprecise", "fuzzy", "wider"), in spite of the approximate reasoning method, where those rules has the dominance, whose consequents are more precise (more "crisp", "narrower"). Basically this is a question of the importance of the vagueness in the rule consequences. Which rule needs more attendance, those, whose consequences are more global, or those, whose consequences are more precise.
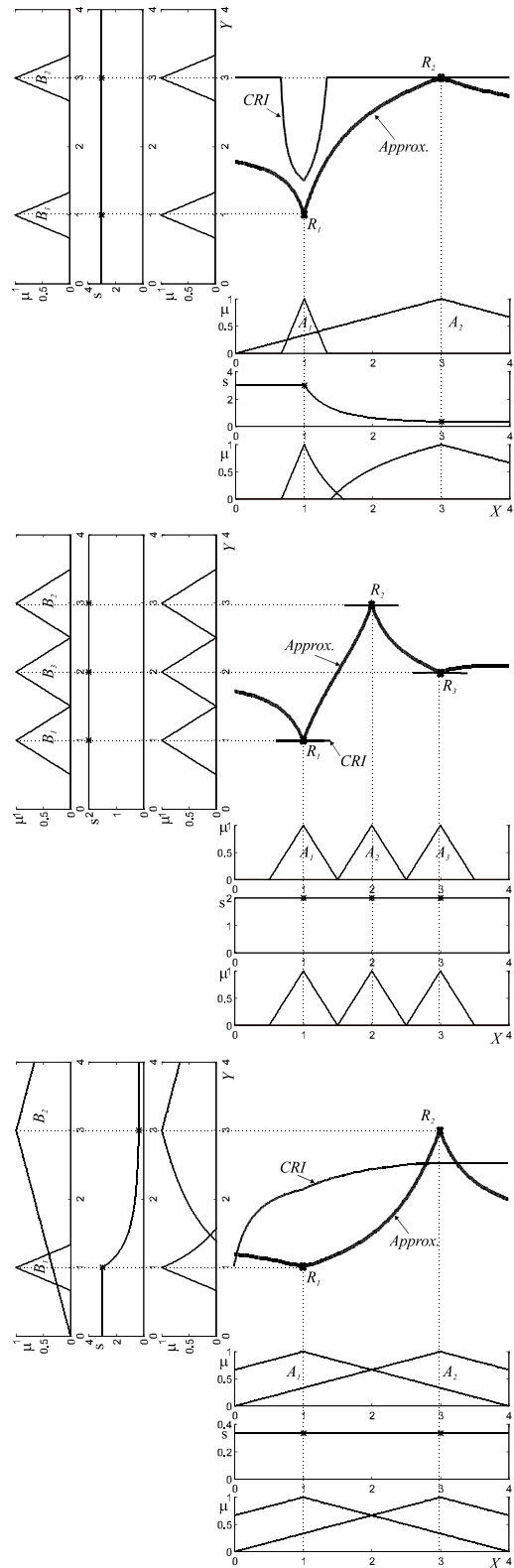

Fig.5. Interpolation of two fuzzy rules (R_i:$A_i{\rightarrow}B_i$) in the approximated vague environment of the fuzzy rulebase, using the proposed rational interpolation (p=1) and the min-max. CRI with the centre of gravity defuzzification

Conclusion

Using the concept of vague environment in most of the practical cases we can built approximate fuzzy reasoning methods simple enough to be a good alternative of the classical Compositional Rule of Inference methods in practical applications.

The advantages (compared to CRI) of the methods proposed in this paper are the following:

- the computational efforts needed for the conclusion can be reduced by reducing the number of the fuzzy rules (the unimportant "filling" rules can be eliminated)

- the proposed method gives conclusion in case of insufficient evidence (sparse fuzzy rulebase) too

- using the proposed approximate fuzzy reasoning methods, if crisp conclusion is needed, it can be fetched directly from the vague conclusion (there are no additional defuzzification step needed)

The vague conclusion calculated by the proposed approximate fuzzy reasoning methods is basically one point. For transforming this point to a fuzzy conclusion, we have to examine the consequence universe. Supposing that the terms in the fuzzy partition of the consequence universe describes all the main properties of the consequence universe and the scaling function approximated from this terms is proper, we can calculate the membership function of the fuzzy conclusion as a level of similarity to the vague conclusion in the vague environment of the consequence universe.

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