**Kovács, Sz.**: *New Aspects of Interpolative Reasoning*, Proceedings of the 6th. International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, pp.477-482, Granada, Spain, (1996). – Draft version.

# New Aspects of Interpolative Reasoning

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**Abstract**

Using the concept of vague environment described by scaling functions [2] instead of the linguistic terms of the fuzzy partition gives a simple way for fuzzy approximate reasoning. In the next steps I would like to introduce a way of fuzzy interpolative reasoning, using the method of interpolation in vague environment.

## 1. Introduction

Widely used way of fuzzy approximate reasoning is based on similarity measures described by distance measure of fuzzy sets [1]. In most of the practical applications, the universes of the fuzzy partitions (used as primary sets of the fuzzy rulebase) can be described by vague environments. Comparing a description of a universe given by a fuzzy partition to the way of using the concept of vague environment we can say, that the linguistic terms of the fuzzy partition are points in the vague environment, while the shape of the fuzzy sets is described by the scaling function. So the similarity measure of fuzzy sets needed for approximate reasoning can be calculated as vague distance of points. This case the primary fuzzy sets of the antecedent and the consequent parts of the fuzzy rules are points in their vague environments, so the fuzzy rules themselves are points in their vague environment too (in the vague environment of the fuzzy rulebase). It means, that the question of approximate fuzzy reasoning can be reduced to the problem of interpolation of the rule points in the vague environment of the fuzzy rulebase relation [4].

## 2. Similarity of fuzzy sets and vague distance of points

The concept of vague environment is based on the similarity or indistinguishability of the elements. Two values in the vague environment are ε-distinguishable if their distance is grater then ε. The distances in vague environment are weighted distances. The weighting factor or function is called scaling function (factor) [2].

Two values in the vague environment *X* are ε-distinguishable if

$$\varepsilon > \delta_s(x_1, x_2) = \left| \int_{x_2}^{x_1} s(x)\,dx \right|$$

where $\delta_s(x_1, x_2)$ is the vague distance of the values $x_1$, $x_2$ and $s(x)$ is the scaling function.

For finding connections between fuzzy sets and a vague environment we can introduce the membership function $\mu_{x0}(x)$ as a level of similarity $x_0$ to $x$, as the degree to which $x$ is indistinguishable to $x_0$ [2]. The α-cuts of the fuzzy set $\mu_{x0}(x)$ is the set which contains the elements that are (1−α)-indistinguishable from $x_0$ (see Fig.1.):

$$\delta_s(\mathrm{a}, \mathrm{b}) \le 1 - \alpha$$

$$\mu_{x0}(x) = 1 - \min\{\delta_s(\mathrm{a}, \mathrm{b}), 1\} = 1 - \min\left\{ \left| \int_{\mathrm{a}}^{\mathrm{b}} s(x)\,dx \right|, 1 \right\}$$

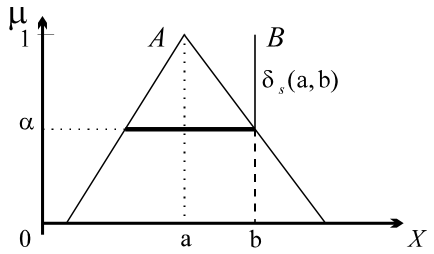

Fig.1.

It is very easy to realise (see Fig.1.), that this case the vague distance of points a and b ($\delta_s(\mathrm{a},\mathrm{b})$) is basically the *Disconsistency Measure* ($S_D$) of the fuzzy sets $A$ and $B$ (where $B$ is a singleton):

$$S_D = 1 - \sup_{x \in X} \mu_{A \cap B}(x) = \delta_s(\mathrm{a},\mathrm{b})$$
$$\text{if} \quad \delta_s(\mathrm{a},\mathrm{b}) \in [0,1]$$

where $A \cap B$ is the min t-norm.

It means, that we can calculate the disconsistency measures between member fuzzy sets of a fuzzy partition and a singleton, as vague distances of points in the vague environment of the fuzzy partition. The difference between the disconsistency measures and the vague distances is, that the vague distances are not limited to [0,1]. That is why they are useful in approximate fuzzy reasoning.
So if it is possible to describe all the fuzzy partitions of the primary fuzzy sets (the antecedent and consequent universes) of our fuzzy rulebase, and the observation is a singleton, we can calculate the disconsistency measures of the antecedent primary fuzzy sets of the rulebase and the observation, and the disconsistency measures of the consequent primary fuzzy sets and the consequence (we are looking for) as vague distances of points in the antecedent and consequent vague universes.

## 3. Generating vague environments from the fuzzy partitions of the linguistic terms of the fuzzy rules

The vague environment is described by its scaling function. For generating a vague environment we have to find an appropriate scaling function, which describes the shapes of all the terms in the fuzzy partition.
The method proposed by Klawonn [2], for choosing the scaling function, gives an exact description of the fuzzy terms after their reconstruction from the scaling function:

$$s(x) = \left|\mu'(x)\right| = \left|\frac{d\mu}{dx}\right|$$

So we always find a scaling function, if we have only one fuzzy set in the fuzzy partition. Usually the fuzzy partition contains more than one fuzzy set, so this method requires some restrictions for the membership functions of the terms [2]:

$$s(x) \text{ exists iff } \min\left\{\mu_i(x), \mu_j(x)\right\} > 0 \;\Rightarrow$$
$$\left|\mu'_i(x)\right| = \left|\mu'_j(x)\right| \quad \forall i,j \in I$$

Generally the above condition is not fulfilling, so the question is how to describe all fuzzy sets of the fuzzy partition with one “universal” scaling function. For this reason we propose to use the **approximative scaling function**.

## 4. The approximative scaling function

The approximative scaling function is an approximation of the scaling functions describes the terms of the fuzzy partition separately. The simplest way of generating the approximative scaling function is the **linear interpolation of the scaling function** between the neighbouring terms.
Supposing that the fuzzy terms are **triangles**, each fuzzy term can be characterised by two constant scaling functions, the scaling factor of the left and the right slope of the triangle. So a triangle shaped fuzzy term can be characterised by three values (by a triple), by the values of the left and the right scaling factors and the value of its core point (e.g.Fig.2.).
Using the method of linear interpolation, we are calculating the approximative scaling function as a piecewise linear function, which is a linear interpolation of the right side

scaling factor of the left neighbouring term and the left side scaling factor of the right neighbouring term (e.g.Fig.3.):

$$s(x)=\left\{\frac{s_{i+1}^{L}-s_{i}^{R}}{x_{i+1}-x_{i}}\cdot\left(x-x_{i}\right)+s_{i}^{R}\;\middle|\;x\in\left[x_{i},x_{i+1}\right),\right.$$
$$\forall\, i\in[1,n-1]$$

where s($x$) is the approximative scaling function, $x_i$ is the core of the i[th] term of the approximated fuzzy partition, $s_i^L$, $s_i^R$ are the left and right side scaling factors of the i[th] triangle shaped term of the approximated fuzzy partition, n is the number of the terms in the approximated fuzzy partition

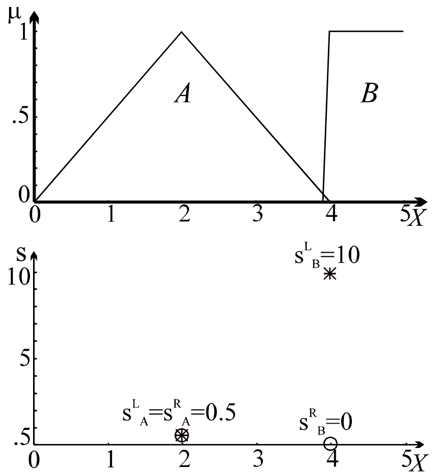


Fig.2. Two triangle shaped fuzzy set characterised by two triple, by the left and the right scaling factors and the value of the core point

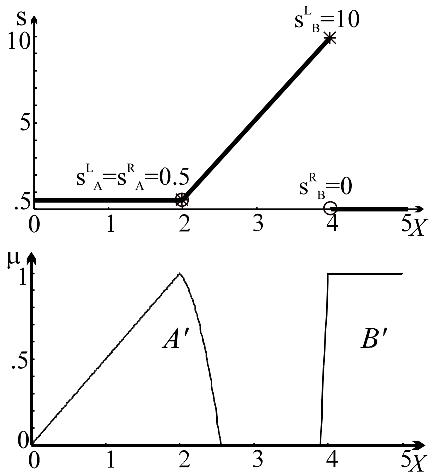


Fig.3. Approximative scaling function generated by linear interpolation of two triangle shaped fuzzy terms (Fig.2.), and these sets as the approximative scaling function describes them (*A'*,*B'*)

The main problem of the linearly interpolated scaling functions, that they cant handle the big differences in neighbouring scaling factors or crisp fuzzy sets correctly. If there are big differences in the neighbouring scaling factors, the bigger scaling factor is "dominating" the smaller factor (see e.g.Fig.2,3). For example if one of the neighbouring fuzzy set is crisp (its scaling factor is infinite), the slope of the linearly interpolated scaling function is infinite too, so both the fuzzy sets described by this scaling function will be crisp.

To solve this dilemma, we suggest to adopt the following non-linear function for interpolating the neighbouring scaling factors:

$$s(x)=\begin{cases}\dfrac{w_i}{(d_i+1)^{k\cdot w_i}-1}\cdot\left(\dfrac{(d_i+1)^{k\cdot w_i}}{(x-x_i+1)^{k\cdot w_i}}-1\right)+s_{i+1}^{L} & \middle|\; s_i^R\ge s_{i+1}^L,\\[2ex] \dfrac{w_i}{(d_i+1)^{k\cdot w_i}-1}\cdot\left(\dfrac{(d_i+1)^{k\cdot w_i}}{(x_{i+1}-x+1)^{k\cdot w_i}}-1\right)+s_{i}^{R} & \middle|\; s_i^R< s_{i+1}^L,\end{cases}$$

$$x\in\left[x_i,x_{i+1}\right),\ \forall\, i\in[1,n-1],$$

where

$$w_i=\left|s_{i+1}^L-s_i^R\right|,\ \forall\, i\in[1,n-1],$$

$$d_i=x_{i+1}-x_i,\ \forall\, i\in[1,n-1],$$

s($x$) is the approximative scaling function, $x_i$ is the core of the i[th] term of the approximated fuzzy partition, $s_i^L$, $s_i^R$ are the left and right side scaling factors of the i[th] triangle shaped term of the approximated fuzzy partition, $k$ constant factor of sensitivity for neighbouring scaling factor differences, n is the number of the terms in the approximated fuzzy partition.

The above function has the following useful properties:

If the neighbouring scaling factors are equals, s($x$)is linear:

$$s_i^R=s_{i+1}^L,\ x\in\left[x_i,x_{i+1}\right)\ \Rightarrow\ s(x)=s_i^R=s_{i+1}^L$$

If one of the neighbouring scaling factors is infinite e.g. $s_i^R\to\infty$ (the right side of the i[th] term is crisp) then:

$$\mathrm{s}_{\mathrm{i}}^{\mathrm{R}} \to \infty \text{ and } \mathrm{s}_{\mathrm{i+1}}^{\mathrm{L}} \text{ finite, } x \in [\mathrm{x}_{\mathrm{i}}, \mathrm{x}_{\mathrm{i+1}})$$

$$\Rightarrow \ \mathrm{s}(x) \to \begin{cases} \infty \mid x = \mathrm{x}_{\mathrm{i}} \\ 0 \text{ otherwise} \end{cases}$$

similarly

$$\mathrm{s}_{\mathrm{i+1}}^{\mathrm{L}} \to \infty \text{ and } \mathrm{s}_{\mathrm{i}}^{\mathrm{R}} \text{ finite, } x \in [\mathrm{x}_{\mathrm{i}}, \mathrm{x}_{\mathrm{i+1}})$$

$$\Rightarrow \ \mathrm{s}(x) \to \begin{cases} \infty \mid x \to \mathrm{x}_{\mathrm{i+1}} \\ 0 \text{ otherwise} \end{cases}$$

Fig.4. and Fig.5. show some examples for the application of the proposed non-linear function.

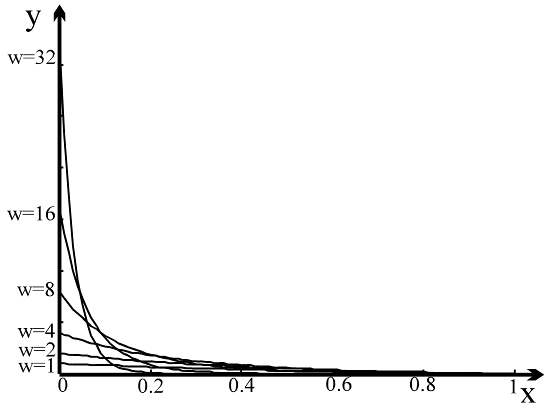


Fig.4. Examples for the proposed non-linear function ($x_1$=0, $x_2$=1, $s_1^R \geq s_2^L$, $k$=1)

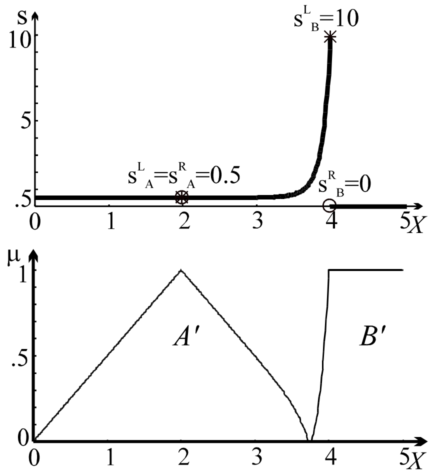


Fig.5. Approximative scaling function generated by the proposed non-linear function ($k$=1), and the original fuzzy partition ($A$,$B$) as this scaling function describes it ($A'$,$B'$)

## 5. Calculating the conclusion by approximating the vague points of the rulebase

If the vague environment of a fuzzy partition (the scaling function or the approximative scaling function) exists, the member sets of the fuzzy partition can be characterised by points in the vague environment. (In our case the points are characterising the cores of the terms, while the shapes of the membership functions are described by the scaling function.) If all the vague environments of the antecedent and consequent universes of the fuzzy rulebase are exist, all the primary fuzzy sets (linguistic terms) used in the fuzzy rulebase can be characterised by points in their vague environment. So the fuzzy rules (build on the primary fuzzy sets) can be characterised by points in the vague environment of the fuzzy rulebase too. This case the approximative fuzzy reasoning can be handled as a classical interpolation task. Applying the concept of vague environment (the distances of points are weighted distances), we can use the classical interpolation methods for approximative fuzzy reasoning.

For example we can adopt the method of **linear rule interpolation of two fuzzy rules** for vague environment. This interpolation method deals only with two rules from rule base, whose antecedents are the closest flanking antecedents to the observation.

The *Fundamental Equation of the Linear Interpolation of Two Fuzzy Rules* is:

$$\mathrm{dist}(\boldsymbol{A}_1,\boldsymbol{x}):\mathrm{dist}(\boldsymbol{x},\boldsymbol{A}_2) = \mathrm{dist}(\boldsymbol{B}_1,\boldsymbol{y}):\mathrm{dist}(\boldsymbol{y},\boldsymbol{B}_2)$$

where $\boldsymbol{A}_1 \langle \boldsymbol{x} \langle \boldsymbol{A}_2$ and $\boldsymbol{B}_1 \langle \boldsymbol{B}_2$ the two fuzzy rules flanks the observation $\boldsymbol{x}$ in sense of ordering $\langle$, $R_i = \boldsymbol{A}_i \to \boldsymbol{B}_i \quad i \in [1,2]$

(The restrictions for the two chosen rules are: the flanking of the observation on the antecedent side and the existence of the ordering on the consequent side. The dist($\boldsymbol{F}$,$\boldsymbol{G}$) denotes the fuzzy distance. See more detailed in [3].)

Without hurting generality we suppose, that the consequence universe of the fuzzy rules is one dimensional (multidimensional case can be decomposed to one dimensional one) and antecedent universe is multidimensional. Substituting the formula of vague distance to the equation of the linear interpolation of two fuzzy rules we get (the multidimensional distances are in Euclidean sense):

$$\sqrt{\sum_{i=1}^{m}\left(\int_{A_{1i}}^{x_i} s_{xi}\left(x_i\right)dx_i\right)^2} : \sqrt{\sum_{i=1}^{m}\left(\int_{x_i}^{A_{2i}} s_{xi}\left(x_i\right)dx_i\right)^2} = \left|\int_{B_1}^{y} s_Y\left(y\right)dy\right| : \left|\int_{y}^{B_2} s_Y\left(y\right)dy\right|$$

where $s_{xi}$ is the i[th] scaling function of the $m$ dimensional antecedent universe, $s_y$ the scaling function of the consequence universe

For one dimensional antecedent universe case see example on Fig.6.

## 6. Generating the fuzzy conclusion described by the point of the vague conclusion

The vague conclusion calculated by rule interpolation is basically one point. For transforming this point to a fuzzy conclusion, we have to examine the consequence universe. Supposing that the terms in the fuzzy partition of the consequence universe describes all the main properties of the consequence universe and the scaling function approximated from this terms is proper, we can calculate the membership function of the fuzzy conclusion as a level of similarity to the vague conclusion in the vague environment of the consequence universe (e.g. Fig.6.):

$$\mu_y(y) = 1 - \min\left\{\delta_{sy}(y, y_v), 1\right\} = 1 - \min\left\{\left|\int_{y}^{y_v} s_y\left(y\right)dy\right|, 1\right\},$$

where $s_y$ is the scaling function of the consequence universe, $y_v$ is the vague conclusion.

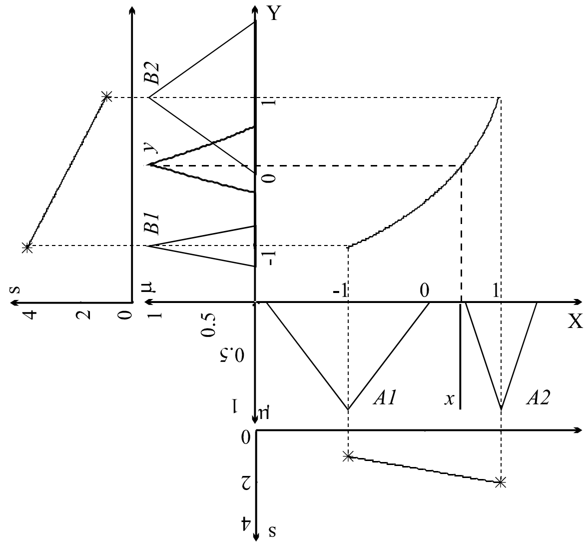


Fig.6. Linear interpolation of two fuzzy rules in approximated vague environment, the approximative scaling functions are generated by linear interpolation of the scaling value points $A_1$, $A_2$, $B_1$, $B_2$, the fuzzy rules are $R_1$: $A_1 \rightarrow B_1$ , $R_2$: $A_2 \rightarrow B_2$

## 7. Extending the method of vague environment based approximative fuzzy reasoning from singleton to fuzzy observation

Using the idea of *equal Disconsistency Measure* we can extend the concept of vague environment from similarity relations between the member sets of a fuzzy partition and singletons to similarity relations between the members of two fuzzy partitions:

$$S_D = 1 - \sup_{x \in X} \mu_{A \cap B}\left(x\right) = \delta_s(\mathrm{a},\mathrm{b}) \text{ if } \delta_s(\mathrm{a},\mathrm{b}) \in \left[0,1\right]$$

We would like to find a *resultant scaling function* $s'_A\left(x\right)$, which unifies the two scaling functions of the two independent fuzzy partitions, generating a fuzzy set, which has the same disconsistency measure to a singleton, as the two fuzzy sets described by the original scaling functions had (see Fig.7.):

$$\delta_s(\mathrm{a},\mathrm{b}) = \left|\int_{\mathrm{a}}^{x_0} s_A\left(x\right)dx\right| = \left|\int_{x_0}^{\mathrm{b}} s_B\left(x\right)dx\right| = \left|\int_{\mathrm{a}}^{\mathrm{b}} s'_A\left(x\right)dx\right|$$

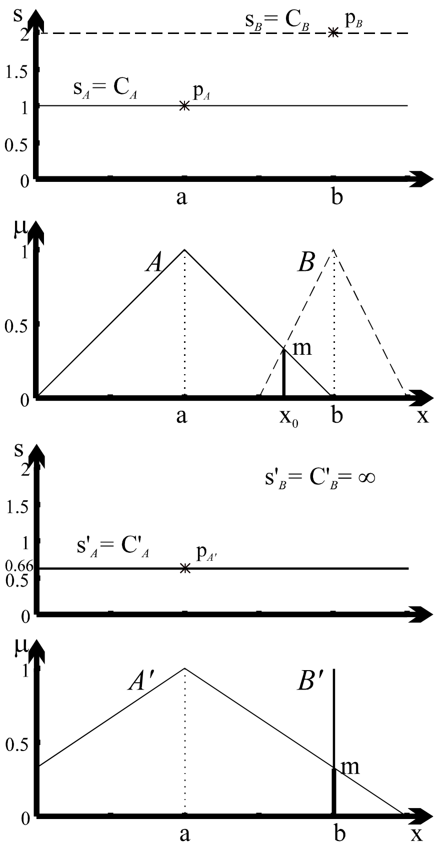


Fig.7. The *resultant scaling function* $s'_A(x)$, has the same disconsistency measure to a singleton, as the two fuzzy sets described by the original scaling functions have

This question is very easy to solve if the *scaling functions are constants* (Fig.7.):

$$\mathrm{m} = 1 - \mathrm{C}_A \cdot (x_0 - \mathrm{a}) = 1 - \mathrm{C}_B \cdot (\mathrm{b} - x_0) =$$

$$= 1 - \frac{\mathrm{C}_A \cdot \mathrm{C}_B}{\mathrm{C}_A + \mathrm{C}_B} \cdot (\mathrm{b} - \mathrm{a}) = 1 - \mathrm{C}'_A \cdot (\mathrm{b} - \mathrm{a})$$

$$\mathrm{C}'_A = \frac{\mathrm{C}_A \cdot \mathrm{C}_B}{\mathrm{C}_A + \mathrm{C}_B} \Rightarrow \quad s'_A(x) = \frac{s_A(x) \cdot s_B(x)}{s_A(x) + s_B(x)}$$

$$\forall \; x \in X$$

It is very easy to proof, that generally the resultant scaling function is not exists, but the result of the constant scaling functions case, as a kind of "approximation", seems to be useful in practical applications. Using the proposed formula ($s'_A(x)$ above) we can unify two independent fuzzy partitions (two independent vague environment) into one, simplifying the question of calculating the disconsistency measure of two arbitrary fuzzy sets to calculating the disconsistency measure of one fuzzy set and a singleton - to calculate the vague distance of two points in a vague environment (e.g.Fig.8.).

In practical sense it means, that if it is possible to describe the universes of observations by vague environments, we can unify these vague environments with the appropriate vague environments of the antecedent universes of the fuzzy rulebase. This way we can transform the method of vague environment based approximative fuzzy reasoning from singleton to fuzzy observation.

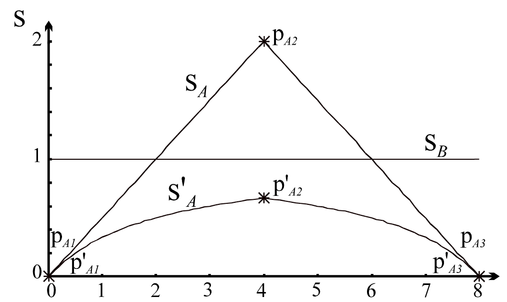


Fig.8. Unifying scaling functions $s_A$, $s_B$ to $s'_A$


**Acknowledgements**

I would like to say thanks to my supervisor **prof. L. T. Kóczy** for his unselfish help.


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